8 ☐ FURTHER APPLICATIONS OF INTEGRATION

8.1 Arc Length

1. $y = 2 - 3x \quad \Rightarrow \quad L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\, dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

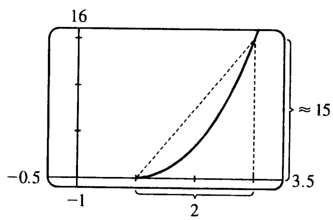
2. Using the arc length formula with $y = \sqrt{4 - x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$, we get

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx = \int_0^2 \frac{2\, dx}{\sqrt{4 - x^2}} = 2 \lim_{t \to 2^-} \int_0^t \frac{dx}{\sqrt{2^2 - x^2}}$$

$$= 2 \lim_{t \to 2^-} \left[\sin^{-1}(x/2)\right]_0^t = 2 \lim_{t \to 2^-} \left[\sin^{-1}(t/2) - \sin^{-1} 0\right] = 2\left(\tfrac{\pi}{2} - 0\right) = \pi$$

The curve is a quarter of a circle with radius 2, so the length of the arc is $\frac{1}{4}(2\pi \cdot 2) = \pi$, as above.

3.



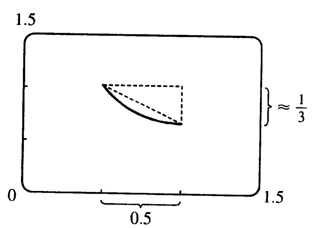
From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5. $y = \frac{2}{3}(x^2 - 1)^{3/2} \quad \Rightarrow \quad y' = (x^2 - 1)^{1/2}(2x) \quad \Rightarrow$

$1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$, so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\, dx = \int_1^3 \left|2x^2 - 1\right| dx = \int_1^3 (2x^2 - 1)\, dx = \left[\tfrac{2}{3}x^3 - x\right]_1^3$$

$$= (18 - 3) - \left(\tfrac{2}{3} - 1\right) = \tfrac{46}{3} = 15.\overline{3}$$

4.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5, f(0.5) \approx 1)$, $(1, f(0.5) \approx 1)$ and $\left(1, \frac{2}{3}\right)$, where $y = f(x) = x^3/6 + 1/(2x)$. This length is about $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} \approx 0.6$, so we might estimate the length to be 0.65.

$$y = \frac{x^3}{6} + \frac{1}{2x} \quad \Rightarrow \quad y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \quad \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$L = \int_{1/2}^{1} \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2}\, dx = \int_{1/2}^{1} \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^{1}$$

$$= \left(\tfrac{1}{6} - \tfrac{1}{2}\right) - \left(\tfrac{1}{48} - 1\right) = \tfrac{31}{48} = 0.6458\overline{3}$$

5. $y = 1 + 6x^{3/2} \Rightarrow dy/dx = 9x^{1/2} \Rightarrow 1 + (dy/dx)^2 = 1 + 81x$. So

$L = \int_0^1 \sqrt{1+81x}\,dx = \int_1^{82} u^{1/2}\left(\frac{1}{81}\,du\right)$ [where $u = 1 + 81x$ and $du = 81\,dx$]

$= \frac{1}{81}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{82} = \frac{2}{243}\left(82\sqrt{82} - 1\right)$

6. $y^2 = 4(x+4)^3,\ y > 0 \Rightarrow y = 2(x+4)^{3/2} \Rightarrow dy/dx = 3(x+4)^{1/2} \Rightarrow$

$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37$. So

$$L = \int_0^2 \sqrt{9x+37}\,dx \quad \begin{bmatrix} u = 9x+37, \\ du = 9\,dx \end{bmatrix} \quad = \int_{37}^{55} u^{1/2}\left(\frac{1}{9}\,du\right)$$

$= \frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right)$

7. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3} \Rightarrow \dfrac{dy}{dx} = \dfrac{5}{6}x^4 - \dfrac{3}{10}x^{-4} \Rightarrow$

$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$. So

$$L = \int_1^2 \sqrt{\left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)^2}\,dx = \int_1^2 \left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)dx = \left[\tfrac{1}{6}x^5 - \tfrac{1}{10}x^{-3}\right]_1^2$$

$= \left(\frac{32}{6} - \frac{1}{80}\right) - \left(\frac{1}{6} - \frac{1}{10}\right) = \frac{31}{6} + \frac{7}{80} = \frac{1261}{240}$

8. $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4} \Rightarrow \dfrac{dy}{dx} = x - \dfrac{1}{4x} \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = x^2 + \dfrac{1}{2} + \dfrac{1}{16x^2}$. So

$$L = \int_2^4 \left(x + \frac{1}{4x}\right)dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}.$$

9. $x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \Rightarrow dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$L = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \frac{1}{2}\left[\left(\frac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\frac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$

$= \frac{1}{2}\left(24 - \frac{8}{3}\right) = \frac{1}{2}\left(\frac{64}{3}\right) = \frac{32}{3}$

10. $y = \ln(\cos x) \Rightarrow dy/dx = -\tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x$. So

$L = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = \ln(2+\sqrt{3}) - \ln(1+0) = \ln(2+\sqrt{3})$.

11. $y = \ln(\sec x) \Rightarrow \dfrac{dy}{dx} = \dfrac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \left[\ln(\sec x + \tan x)\right]_0^{\pi/4}$

$= \ln(\sqrt{2}+1) - \ln(1+0) = \ln(\sqrt{2}+1)$

12. $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{1+x^2}}{x}$. So $L = \displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\,dx$. Now

let $v = \sqrt{1+x^2}$, so $v^2 = 1 + x^2$ and $v\,dv = x\,dx$. Thus

$$L = \int_{\sqrt{2}}^2 \frac{v}{v^2-1}v\,dv = \int_{\sqrt{2}}^2 \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right)dv = \left[v + \tfrac{1}{2}\ln|v-1| - \tfrac{1}{2}\ln|v+1|\right]_{\sqrt{2}}^2$$

$$= \left[v - \frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right]_{\sqrt{2}}^2 = 2 - \tfrac{1}{2}\ln 3 - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2 - \sqrt{2} + \ln\left(\sqrt{2}+1\right) - \tfrac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

13. $y = \cosh x \;\Rightarrow\; y' = \sinh x \;\Rightarrow\; 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$.
So $L = \int_0^1 \cosh x\, dx = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}(e - 1/e)$.

14. $y^2 = 4x,\ x = \frac{1}{4}y^2 \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{2}y \;\Rightarrow\; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^2$. So

$$L = \int_0^2 \sqrt{1 + \tfrac{1}{4}y^2}\, dy = \int_0^1 \sqrt{1 + u^2}\cdot 2\, du \qquad [u = \tfrac{1}{2}y,\ dy = 2\, du]$$
$$\overset{21}{=} \left[u\sqrt{1+u^2} + \ln\left|u + \sqrt{1+u^2}\right|\right]_0^1 = \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$$

15. $y = e^x \;\Rightarrow\; y' = e^x \;\Rightarrow\; 1 + (y')^2 = 1 + e^{2x}$. So

$$L = \int_0^1 \sqrt{1 + e^{2x}}\, dx = \int_1^e \sqrt{1+u^2}\,\frac{du}{u} \qquad [u = e^x, \text{ so } x = \ln u,\ dx = du/u]$$
$$= \int_1^e \frac{\sqrt{1+u^2}}{u^2}\, u\, du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}\, v\, dv \qquad \left[v = \sqrt{1+u^2}, \text{ so } v^2 = 1 + u^2,\ v\, dv = u\, du\right]$$
$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv = \left[v + \frac{1}{2}\ln\frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$
$$= \sqrt{1+e^2} + \frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}$$
$$= \sqrt{1+e^2} - \sqrt{2} + \ln\left(\sqrt{1+e^2} - 1\right) - 1 - \ln\left(\sqrt{2} - 1\right)$$

Or: Use Formula 23 for $\int \left(\sqrt{1+u^2}/u\right) du$, or substitute $u = \tan\theta$.

16. $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right) = \ln(e^x+1) - \ln(e^x-1) \;\Rightarrow\; y' = \dfrac{e^x}{e^x+1} - \dfrac{e^x}{e^x-1} = \dfrac{-2e^x}{e^{2x}-1} \;\Rightarrow$

$1 + (y')^2 = 1 + \dfrac{4e^{2x}}{(e^{2x}-1)^2} = \dfrac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2} \;\Rightarrow\; \sqrt{1 + (y')^2} = \dfrac{e^{2x}+1}{e^{2x}-1} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{\cosh x}{\sinh x}$.

So $L = \displaystyle\int_a^b \frac{\cosh x}{\sinh x}\, dx = \Big[\ln\sinh x\Big]_a^b = \ln\sinh b - \ln\sinh a = \ln\left(\frac{\sinh b}{\sinh a}\right) = \ln\left(\frac{e^b - e^{-b}}{e^a - e^{-a}}\right)$.

17. $y = \cos x \;\Rightarrow\; dy/dx = -\sin x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1 + \sin^2 x}\, dx$.

18. $y = 2^x \;\Rightarrow\; dy/dx = (2^x)\ln 2 \;\Rightarrow\; L = \int_0^3 \sqrt{1 + (\ln 2)^2\, 2^{2x}}\, dx$

19. $x = y + y^3 \;\Rightarrow\; dx/dy = 1 + 3y^2 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + (1 + 3y^2)^2 = 9y^4 + 6y^2 + 2$.
So $L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy$.

20. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ y = \pm b\sqrt{1 - x^2/a^2} = \pm\dfrac{b}{a}\sqrt{a^2 - x^2}$ [assume $a > 0$].

$y = \dfrac{b}{a}\sqrt{a^2 - x^2} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{-bx}{a\sqrt{a^2 - x^2}} \;\Rightarrow\; \left(\dfrac{dy}{dx}\right)^2 = \dfrac{b^2x^2}{a^2(a^2 - x^2)}$.

So $L = 2\displaystyle\int_{-a}^{a} \left[1 + \frac{b^2x^2}{a^2(a^2 - x^2)}\right]^{1/2} dx = \frac{4}{a}\int_0^a \left[\frac{(b^2 - a^2)x^2 + a^4}{a^2 - x^2}\right]^{1/2} dx$.

21. $y = xe^{-x} \Rightarrow dy/dx = e^{-x} - xe^{-x} = e^{-x}(1-x) \Rightarrow 1 + (dy/dx)^2 = 1 + e^{-2x}(1-x)^2$. Let $f(x) = \sqrt{1+(dy/dx)^2} = \sqrt{1+e^{-2x}(1-x)^2}$. Then $L = \int_0^5 f(x)\,dx$. Since $n = 10$, $\Delta x = \frac{5-0}{10} = \frac{1}{2}$. Now

$$L \approx S_{10} = \tfrac{1/2}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + 2f(3) + 4f\left(\tfrac{7}{2}\right) + 2f(4) + 4f\left(\tfrac{9}{2}\right) + f(5)\right] \approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

22. $x = y + \sqrt{y} \Rightarrow dx/dy = 1 + \dfrac{1}{2\sqrt{y}} \Rightarrow 1 + (dx/dy)^2 = 1 + \left(1 + \dfrac{1}{2\sqrt{y}}\right)^2 = 2 + \dfrac{1}{\sqrt{y}} + \dfrac{1}{4y}$.

Let $g(y) = \sqrt{1+(dx/dy)^2}$. Then $L = \int_1^2 g(y)\,dy$. Since $n = 10$, $\Delta y = \frac{2-1}{10} = \frac{1}{10}$. Now

$$L \approx S_{10} = \tfrac{1/10}{3}[g(1) + 4g(1.1) + 2g(1.2) + 4g(1.3) + 2g(1.4) + 4g(1.5) + 2g(1.6) + 4g(1.7) + 2g(1.8) + 4g(1.9) + g(2)] \approx 1.732215,$$

which is the same value of the integral produced by a calculator to six decimal places.

23. $y = \sec x \Rightarrow dy/dx = \sec x \tan x \Rightarrow L = \int_0^{\pi/3} f(x)\,dx$, where $f(x) = \sqrt{1 + \sec^2 x \tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Now

$$L \approx S_{10} = \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + 4f\left(\frac{3\pi}{30}\right) + 2f\left(\frac{4\pi}{30}\right) + 4f\left(\frac{5\pi}{30}\right) + 2f\left(\frac{6\pi}{30}\right) + 4f\left(\frac{7\pi}{30}\right) + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 1.569619.$$

The value of the integral produced by a calculator is 1.569259 (to six decimal places).

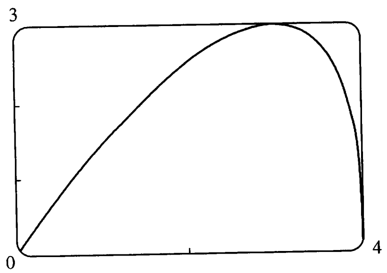
24. $y = x\ln x \Rightarrow dy/dx = 1 + \ln x$. Let $f(x) = \sqrt{1+(dy/dx)^2} = \sqrt{1+(1+\ln x)^2}$.

Then $L = \int_1^3 f(x)\,dx$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Now

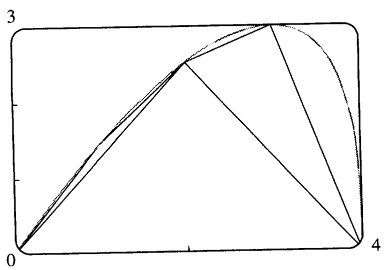
$$L \approx S_{10} = \tfrac{1/5}{3}[f(1) + 4f(1.2) + 2f(1.4) + 4f(1.6) + 2f(1.8) + 4f(2) + 2f(2.2) + 4f(2.4) + 2f(2.6) + 4f(2.8) + f(3)] \approx 3.869618.$$

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

25. (a)



(b)



Let $f(x) = y = x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0,0)$ and $(4, f(4)) = (4, 0)$, and its length is 4. The polygon with two sides joins the points $(0,0)$, $(2, f(2)) = (2, 2\sqrt[3]{2})$ and $(4,0)$.

Its length is

$$\sqrt{(2-0)^2+\left(2\sqrt[3]{2}-0\right)^2}+\sqrt{(4-2)^2+\left(0-2\sqrt[3]{2}\right)^2}=2\sqrt{4+2^{8/3}}\approx 6.43$$

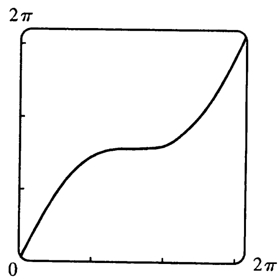
Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1,\sqrt[3]{3}\right)$, $\left(2,2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length is

$$\sqrt{1+\left(\sqrt[3]{3}\right)^2}+\sqrt{1+\left(2\sqrt[3]{2}-\sqrt[3]{3}\right)^2}+\sqrt{1+\left(3-2\sqrt[3]{2}\right)^2}+\sqrt{1+9}\approx 7.50$$

(c) Using the arc length formula with $\dfrac{dy}{dx}=x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right]+\sqrt[3]{4-x}=\dfrac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is $L=\displaystyle\int_0^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^4\sqrt{1+\left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx.$

(d) According to a CAS, the length of the curve is $L\approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

26. (a) Let $f(x)=y=x+\sin x$ with $0\le x\le 2\pi$.

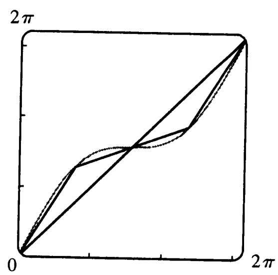


(b) The polygon with one side is just the line segment joining the points $(0,f(0))=(0,0)$ and $(2\pi,f(2\pi))=(2\pi,2\pi)$, and its length is $\sqrt{(2\pi-0)^2+(2\pi-0)^2}=2\sqrt{2}\,\pi\approx 8.9$.

The polygon with two sides joins the points $(0,0)$, $(\pi,f(\pi))=(\pi,\pi)$, and $(2\pi,2\pi)$. Its length is

$$\sqrt{(\pi-0)^2+(\pi-0)^2}+\sqrt{(2\pi-\pi)^2+(2\pi-\pi)^2}=\sqrt{2}\,\pi+\sqrt{2}\,\pi=2\sqrt{2}\,\pi\approx 8.9$$

Note from the diagram that the two approximations are the same because the sides of the 2-sided polygon are in fact on the same line, since $f(\pi)=\pi=\frac{1}{2}f(2\pi)$.



The four-sided polygon joins the points $(0,0)$, $\left(\frac{\pi}{2},\frac{\pi}{2}+1\right)$, (π,π), $\left(\frac{3\pi}{2},\frac{3\pi}{2}-1\right)$, and $(2\pi,2\pi)$, so its length is

$$\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}+1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}-1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}-1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}+1\right)^2}\approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1 + \cos x$, the length of the curve is

$$L = \int_0^{2\pi} \sqrt{1 + (1 + \cos x)^2}\, dx = \int_0^{2\pi} \sqrt{2 + 2\cos x + \cos^2 x}\, dx$$

(d) The CAS approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

27. $x = \ln\left(1 - y^2\right) \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{-2y}{1 - y^2} \;\Rightarrow\; 1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \dfrac{4y^2}{(1 - y^2)^2} = \dfrac{\left(1 + y^2\right)^2}{\left(1 - y^2\right)^2}$. So

$$L = \int_0^{1/2} \sqrt{\frac{\left(1 + y^2\right)^2}{\left(1 - y^2\right)^2}}\, dy = \int_0^{1/2} \frac{1 + y^2}{1 - y^2}\, dy = \ln 3 - \tfrac{1}{2} \text{ [from a CAS] } \approx 0.599$$

28. $y = x^{4/3} \;\Rightarrow\; dy/dx = \frac{4}{3}x^{1/3} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \frac{16}{9}x^{2/3} \;\Rightarrow$

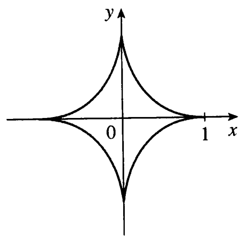
$$L = \int_0^1 \sqrt{1 + \tfrac{16}{9}x^{2/3}}\, dx = \int_0^{4/3} \sqrt{1 + u^2}\, \tfrac{81}{64}u^2\, du \qquad \begin{bmatrix} u = \frac{4}{3}x^{1/3},\; du = \frac{4}{9}x^{-2/3}\, dx, \\ dx = \frac{9}{4}x^{2/3}\, du = \frac{9}{4} \cdot \frac{9}{16}u^2\, du = \frac{81}{64}u^2\, du \end{bmatrix}$$

$$\overset{22}{=} \tfrac{81}{64}\left[\tfrac{1}{8}u(1 + 2u^2)\sqrt{1 + u^2} - \tfrac{1}{8}\ln\left(u + \sqrt{1 + u^2}\,\right)\right]_0^{4/3}$$

$$= \tfrac{81}{64}\left[\tfrac{1}{6}\left(1 + \tfrac{32}{9}\right)\sqrt{\tfrac{25}{9}} - \tfrac{1}{8}\ln\left(\tfrac{4}{3} + \sqrt{\tfrac{25}{9}}\right)\right] = \tfrac{81}{64}\left(\tfrac{1}{6} \cdot \tfrac{41}{9} \cdot \tfrac{5}{3} - \tfrac{1}{8}\ln 3\right)$$

$$= \tfrac{205}{128} - \tfrac{81}{512}\ln 3 \approx 1.4277586$$

29. $y^{2/3} = 1 - x^{2/3} \;\Rightarrow\; y = \left(1 - x^{2/3}\right)^{3/2} \;\Rightarrow$

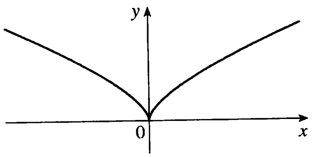


$$\frac{dy}{dx} = \tfrac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\tfrac{2}{3}x^{-1/3}\right) = -x^{-1/3}\left(1 - x^{2/3}\right)^{1/2} \;\Rightarrow$$

$\left(\dfrac{dy}{dx}\right)^2 = x^{-2/3}\left(1 - x^{2/3}\right) = x^{-2/3} - 1$. Thus

$$L = 4\int_0^1 \sqrt{1 + (x^{-2/3} - 1)}\, dx = 4\int_0^1 x^{-1/3}\, dx = 4\lim_{t \to 0^+}\left[\tfrac{3}{2}x^{2/3}\right]_t^1 = 6.$$

30. (a)



(b) $y = x^{2/3} \;\Rightarrow\; 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx$

[an improper integral]. $x = y^{3/2} \;\Rightarrow\; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy$.

The second integral equals $\frac{4}{9} \cdot \frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13} - 8}{27}$. The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\, dx = \lim_{t \to 0^+}\int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx = \lim_{t \to 0^+}\int_{9t^{2/3}}^9 \frac{\sqrt{u + 4}}{18}\, du \qquad \begin{bmatrix} u = 9x^{2/3}, \\ du = 6x^{-1/3}\, dx \end{bmatrix}$$

$$= \int_0^9 \frac{\sqrt{u + 4}}{18}\, du = \frac{1}{18} \cdot \left[\frac{2}{3}(u + 4)^{3/2}\right]_0^9 = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{13\sqrt{13} - 8}{27}$$

(c) L = length of the arc of this curve from $(-1, 1)$ to $(8, 4)$

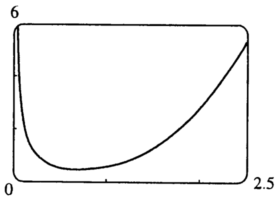
$$= \int_0^1 \sqrt{1 + \tfrac{9}{4}y}\,dy + \int_0^4 \sqrt{1 + \tfrac{9}{4}y}\,dy = \tfrac{13\sqrt{13} - 8}{27} + \tfrac{8}{27}\left[\left(1 + \tfrac{9}{4}y\right)^{3/2}\right]_0^4 \qquad \text{[from part (b)]}$$

$$= \tfrac{13\sqrt{13} - 8}{27} + \tfrac{8}{27}\left(10\sqrt{10} - 1\right) = \tfrac{13\sqrt{13} + 80\sqrt{10} - 16}{27}$$

31. $y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1 + (y')^2 = 1 + 9x$. The arc length function with starting point $P_0(1, 2)$ is

$$s(x) = \int_1^x \sqrt{1 + 9t}\,dt = \left[\tfrac{2}{27}(1 + 9t)^{3/2}\right]_1^x = \tfrac{2}{27}\left[(1 + 9x)^{3/2} - 10\sqrt{10}\right]$$

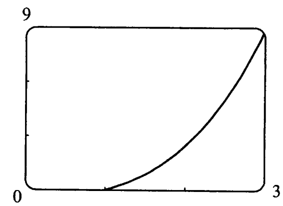
32. (a)



(b) $1 + \left(\dfrac{dy}{dx}\right)^2 = x^4 + \dfrac{1}{2} + \dfrac{1}{16x^4}$,

$$s(x) = \int_1^x \left[t^2 + 1/(4t^2)\right] dt$$
$$= \left[\tfrac{1}{3}t^3 - 1/(4t)\right]_1^x$$
$$= \tfrac{1}{3}x^3 - 1/(4x) - \left(\tfrac{1}{3} - \tfrac{1}{4}\right)$$
$$= \tfrac{1}{3}x^3 - 1/(4x) - \tfrac{1}{12} \quad \text{for } x \geq 1$$

(c)



33. The prey hits the ground when $y = 0 \Leftrightarrow 180 - \frac{1}{45}x^2 = 0 \Leftrightarrow x^2 = 45 \cdot 180 \Rightarrow x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \Rightarrow 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \tfrac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1 + u^2}\left(\tfrac{45}{2}\,du\right) \qquad \left[u = \tfrac{2}{45}x,\ du = \tfrac{2}{45}\,dx\right]$$

$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^4$$

$$= \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4 + \sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4 + \sqrt{17}\right) \approx 209.1 \text{ m}$$

34. $y = 150 - \frac{1}{40}(x - 50)^2 \Rightarrow y' = -\frac{1}{20}(x - 50) \Rightarrow 1 + (y')^2 = 1 + \frac{1}{20^2}(x - 50)^2$, so the distance traveled by the kite is

$$L = \int_0^{80} \sqrt{1 + \tfrac{1}{20^2}(x - 50)^2}\,dx = \int_{-5/2}^{3/2} \sqrt{1 + u^2}\,(20\,du) \qquad \left[u = \tfrac{1}{20}(x - 50),\ du = \tfrac{1}{20}\,dx\right]$$

$$\overset{21}{=} 20\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_{-5/2}^{3/2}$$

$$= 10\left[\tfrac{3}{2}\sqrt{\tfrac{13}{4}} + \ln\left(\tfrac{3}{2} + \sqrt{\tfrac{13}{4}}\right) + \tfrac{5}{2}\sqrt{\tfrac{29}{4}} - \ln\left(-\tfrac{5}{2} + \sqrt{\tfrac{29}{4}}\right)\right]$$

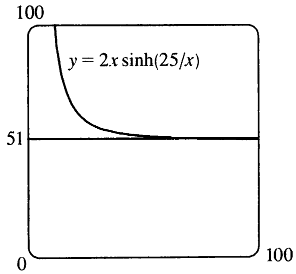
$$= \tfrac{15}{2}\sqrt{13} + \tfrac{25}{2}\sqrt{29} + 10\ln\left(\tfrac{3 + \sqrt{13}}{-5 + \sqrt{29}}\right) \approx 122.8 \text{ ft}$$

35. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x \doteq 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$: $L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx = 2\int_0^{14} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

36. (a) $y = c + a\cosh\left(\frac{x}{a}\right) \Rightarrow y' = \sinh\left(\frac{x}{a}\right) \Rightarrow 1 + (y')^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

$$L = \int_{-b}^{b} \sqrt{\cosh^2\left(\tfrac{x}{a}\right)}\, dx = 2\int_0^b \cosh\left(\tfrac{x}{a}\right) dx = 2\left[a\sinh\left(\tfrac{x}{a}\right)\right]_0^b = 2a\sinh\left(\tfrac{b}{a}\right)$$

(b) At $x = 0$, $y = c + a$, so $c + a = 20$. The poles are 50 ft apart, so $b = 25$, and $L = 51 \Rightarrow 51 = 2a\sinh(b/a)$ [from part (a)]. From the figure, we see that $y = 51$ intersects $y = 2x\sinh(25/x)$ at $x \approx 72.3843$ for $x > 0$. So $a \approx 72.3843$ and the wire should be attached at a distance of $y = c + a\cosh(25/a) = 20 - a + a\cosh(25/a) \approx 24.36$ ft above the ground.



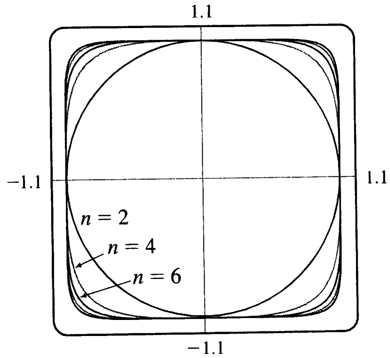
37. $y = \int_1^x \sqrt{t^3 - 1}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{x^3 - 1}$ [by FTC1] $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\sqrt{x^3-1}\right)^2 = x^3 \Rightarrow$

$$L = \int_1^4 \sqrt{x^3}\, dx = \int_1^4 x^{3/2}\, dx = \tfrac{2}{5}\left[x^{5/2}\right]_1^4 = \tfrac{2}{5}(32 - 1) = \tfrac{62}{5} = 12.4$$

38. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.

$x^{2k} + y^{2k} = 1 \Rightarrow y^{2k} = 1 - x^{2k} \Rightarrow y = \left(1 - x^{2k}\right)^{1/(2k)}$ (in the first quadrant), so we use the arc length formula with

$$\frac{dy}{dx} = \frac{1}{2k}\left(1 - x^{2k}\right)^{1/(2k)-1}\left(-2kx^{2k-1}\right)$$
$$= -x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}$$



The total length is therefore

$$L_{2k} = 4\int_0^1 \sqrt{1 + \left[-x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}\right]^2}\, dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}\left(1 - x^{2k}\right)^{1/k-2}}\, dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty}\left(1 - x^{2k}\right)^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

DISCOVERY PROJECT Arc Length Contest

For advice on how to run the contest and a list of student entries, see the article "Arc Length Contest" by Larry Riddle in *The College Mathematics Journal*, Volume 29, No. 4, September 1998, pages 314–320.

8.2 Area of a Surface of Revolution

1. $y = \ln x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1/x)^2}\,dx \;\Rightarrow\; S = \int_1^3 2\pi(\ln x)\sqrt{1 + (1/x)^2}\,dx$ [by (7)]

2. $y = \sin^2 x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (2\sin x\cos x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/2} 2\pi \sin^2 x\sqrt{1 + (2\sin x\cos x)^2}\,dx$ [by (7)]

3. $y = \sec x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (\sec x\tan x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/4} 2\pi x\sqrt{1 + (\sec x\tan x)^2}\,dx$ [by (8)]

4. $y = e^x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + e^{2x}}\,dx \;\Rightarrow\; S = \int_0^{\ln 2} 2\pi x\sqrt{1 + e^{2x}}\,dx$ [by (8)] or

$\int_1^2 2\pi(\ln y)\sqrt{1 + (1/y)^2}\,dy$ [by (6)]

5. $y = x^3 \;\Rightarrow\; y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx]$$

$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\Big[\tfrac{2}{3}u^{3/2}\Big]_1^{145} = \tfrac{\pi}{27}\big(145\sqrt{145} - 1\big)$$

6. The curve $9x = y^2 + 18$ is symmetric about the x-axis, so we only use its top half, given by $y = 3\sqrt{x - 2}$. $dy/dx = \dfrac{3}{2\sqrt{x - 2}}$, so $1 + (dy/dx)^2 = 1 + \dfrac{9}{4(x - 2)}$. Thus,

$$S = \int_2^6 2\pi \cdot 3\sqrt{x - 2}\sqrt{1 + \frac{9}{4(x - 2)}}\,dx = 6\pi\int_2^6\sqrt{x - 2 + \tfrac{9}{4}}\,dx = 6\pi\int_2^6\left(x + \tfrac{1}{4}\right)^{1/2}dx$$

$$= 6\pi\cdot\tfrac{2}{3}\Big[\left(x + \tfrac{1}{4}\right)^{3/2}\Big]_2^6 = 4\pi\Big[\left(\tfrac{25}{4}\right)^{3/2} - \left(\tfrac{9}{4}\right)^{3/2}\Big] = 4\pi\left(\tfrac{125}{8} - \tfrac{27}{8}\right) = 4\pi\cdot\tfrac{98}{8} = 49\pi$$

7. $y = \sqrt{x} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + [1/(2\sqrt{x}\,)]^2 = 1 + 1/(4x)$. So

$$S = \int_4^9 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_4^9 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = 2\pi\int_4^9\sqrt{x + \tfrac{1}{4}}\,dx$$

$$= 2\pi\Big[\tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\Big]_4^9 = \tfrac{4\pi}{3}\Big[\tfrac{1}{8}(4x + 1)^{3/2}\Big]_4^9 = \tfrac{\pi}{6}\Big(37\sqrt{37} - 17\sqrt{17}\Big)$$

8. $y = \cos 2x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-2\sin 2x)^2}\,dx \;\Rightarrow$

$$S = \int_0^{\pi/6} 2\pi\cos 2x\sqrt{1 + 4\sin^2 2x}\,dx = 2\pi\int_0^{\sqrt{3}}\sqrt{1 + u^2}\left(\tfrac{1}{4}\,du\right) \qquad [u = 2\sin 2x,\ du = 4\cos 2x\,dx]$$

$$\overset{21}{=} \tfrac{\pi}{2}\Big[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\big(u + \sqrt{1 + u^2}\,\big)\Big]_0^{\sqrt{3}} = \tfrac{\pi}{2}\Big[\tfrac{\sqrt{3}}{2}\cdot 2 + \tfrac{1}{2}\ln\big(\sqrt{3} + 2\big)\Big] = \tfrac{\pi\sqrt{3}}{2} + \tfrac{\pi}{4}\ln\big(2 + \sqrt{3}\,\big)$$

9. $y = \cosh x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$S = 2\pi \int_0^1 \cosh x \cosh x \, dx = 2\pi \int_0^1 \tfrac{1}{2}(1 + \cosh 2x)\, dx = \pi\left[x + \tfrac{1}{2}\sinh 2x\right]_0^1$$

$$= \pi\left(1 + \tfrac{1}{2}\sinh 2\right) \quad \text{or} \quad \pi\left[1 + \tfrac{1}{4}\left(e^2 - e^{-2}\right)\right]$$

10. $y = \dfrac{x^3}{6} + \dfrac{1}{2x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x^2}{2} - \dfrac{1}{2x^2} \quad \Rightarrow$

$$\sqrt{1 + (dy/dx)^2} = \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2} \quad \Rightarrow$$

$$S = \int_{1/2}^1 2\pi\left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = 2\pi \int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi \int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4}\right) dx = 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8}\right]_{1/2}^1$$

$$= 2\pi\left[\left(\tfrac{1}{72} + \tfrac{1}{6} - \tfrac{1}{8}\right) - \left(\tfrac{1}{64 \cdot 72} + \tfrac{1}{24} - \tfrac{1}{2}\right)\right] = 2\pi\left(\tfrac{263}{512}\right) = \tfrac{263}{256}\pi$$

11. $x = \frac{1}{3}\left(y^2 + 2\right)^{3/2} \quad \Rightarrow \quad dx/dy = \frac{1}{2}\left(y^2 + 2\right)^{1/2}(2y) = y\sqrt{y^2 + 2} \quad \Rightarrow$

$1 + (dx/dy)^2 = 1 + y^2\left(y^2 + 2\right) = \left(y^2 + 1\right)^2$. So

$$S = 2\pi \int_1^2 y\left(y^2 + 1\right) dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}$$

12. $x = 1 + 2y^2 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$. So

$$S = 2\pi \int_1^2 y\sqrt{1 + 16y^2}\, dy = \tfrac{\pi}{16}\int_1^2 \left(16y^2 + 1\right)^{1/2} 32y\, dy = \tfrac{\pi}{16}\left[\tfrac{2}{3}\left(16y^2 + 1\right)^{3/2}\right]_1^2$$

$$= \tfrac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$$

13. $y = \sqrt[3]{x} \quad \Rightarrow \quad x = y^3 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi \int_1^2 x\sqrt{1 + (dx/dy)^2}\, dy = 2\pi \int_1^2 y^3\sqrt{1 + 9y^4}\, dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1 + 9y^4}\, 36y^3\, dy$$

$$= \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1 + 9y^4\right)^{3/2}\right]_1^2 = \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

14. $y = 1 - x^2 \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 4x^2 \quad \Rightarrow$

$$S = 2\pi \int_0^1 x\sqrt{1 + 4x^2}\, dx = \tfrac{\pi}{4}\int_0^1 8x\sqrt{4x^2 + 1}\, dx = \tfrac{\pi}{4}\left[\tfrac{2}{3}\left(4x^2 + 1\right)^{3/2}\right]_0^1 = \tfrac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

15. $x = \sqrt{a^2 - y^2} \quad \Rightarrow \quad dx/dy = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \quad \Rightarrow$

$$1 + (dx/dy)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \quad \Rightarrow$$

$S = \displaystyle\int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\frac{a}{\sqrt{a^2 - y^2}}\, dy = 2\pi \int_0^{a/2} a\, dy = 2\pi a\left[y\right]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2$. Note that this is

$\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

16. $x = a\cosh(y/a) \Rightarrow 1 + (dx/dy)^2 = 1 + \sinh^2(y/a) = \cosh^2(y/a)$. So

$$S = 2\pi \int_{-a}^{a} a\cosh\left(\frac{y}{a}\right)\cosh\left(\frac{y}{a}\right) dy = 4\pi a \int_0^a \cosh^2\left(\frac{y}{a}\right) dy = 2\pi a \int_0^a \left[1 + \cosh\left(\frac{2y}{a}\right)\right] dy$$

$$= 2\pi a\left[y + \frac{a}{2}\sinh\left(\frac{2y}{a}\right)\right]_0^a = 2\pi a\left[a + \frac{a}{2}\sinh 2\right] = 2\pi a^2\left[1 + \tfrac{1}{2}\sinh 2\right] \text{ or } \frac{\pi a^2(e^2 + 4 - e^{-2})}{2}$$

17. $y = \ln x \Rightarrow dy/dx = 1/x \Rightarrow 1 + (dy/dx)^2 = 1 + 1/x^2 \Rightarrow S = \int_1^3 2\pi \ln x\sqrt{1 + 1/x^2}\, dx$.

Let $f(x) = \ln x\sqrt{1 + 1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3}[f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)] \approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y = x + \sqrt{x} \Rightarrow dy/dx = 1 + \frac{1}{2}x^{-1/2} \Rightarrow 1 + (dy/dx)^2 = 2 + x^{-1/2} + \frac{1}{4}x^{-1} \Rightarrow$

$S = \displaystyle\int_1^2 2\pi(x + \sqrt{x})\sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}\, dx$. Let $f(x) = (x + \sqrt{x})\sqrt{2 + \dfrac{1}{\sqrt{x}} + \dfrac{1}{4x}}$.

Since $n = 10$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \approx 29.506566$.

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

19. $y = \sec x \Rightarrow dy/dx = \sec x\tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \sec^2 x\tan^2 x \Rightarrow$

$S = \int_0^{\pi/3} 2\pi \sec x\sqrt{1 + \sec^2 x\tan^2 x}\, dx$. Let $f(x) = \sec x\sqrt{1 + \sec^2 x\tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + \cdots + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 13.527296.$$

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. $y = (1 + e^x)^{1/2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}(1 + e^x)^{-1/2} \cdot e^x = \dfrac{e^x}{2(1 + e^x)^{1/2}} \Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)} = \frac{(e^x + 2)^2}{4(1 + e^x)} \Rightarrow$$

$$S = \int_0^1 2\pi\sqrt{1 + e^x}\,\frac{e^x + 2}{2\sqrt{1 + e^x}}\, dx = \pi\int_0^1 (e^x + 2)\, dx = \pi\left[e^x + 2x\right]_0^1 = \pi[(e + 2) - (1 + 0)] = \pi(e + 1).$$

Let $f(x) = \frac{1}{2}(e^x + 2)$. Since $n = 10$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \approx 11.681330$.

The value of the integral produced by a calculator is 11.681327 (to six decimal places).

21. $y = 1/x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-1/x^2)^2}\,dx = \sqrt{1 + 1/x^4}\,dx \;\Rightarrow$

$$S = \int_1^2 2\pi \cdot \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi \int_1^2 \frac{\sqrt{x^4+1}}{x^3}\,dx = 2\pi \int_1^4 \frac{\sqrt{u^2+1}}{u^2}\left(\tfrac{1}{2}\,du\right) \qquad [u = x^2,\ du = 2x\,dx]$$

$$= \pi \int_1^4 \frac{\sqrt{1+u^2}}{u^2}\,du \overset{24}{=} \pi\left[-\frac{\sqrt{1+u^2}}{u} + \ln\left(u + \sqrt{1+u^2}\right)\right]_1^4$$

$$= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4+\sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1+\sqrt{2}\right)\right] = \pi\left[\sqrt{2} - \tfrac{\sqrt{17}}{4} + \ln\left(\tfrac{4+\sqrt{17}}{1+\sqrt{2}}\right)\right]$$

22. $y = \sqrt{x^2+1} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}} \;\Rightarrow\; ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \dfrac{x^2}{x^2+1}}\,dx \;\Rightarrow$

$$S = \int_0^3 2\pi\sqrt{x^2+1}\sqrt{1 + \frac{x^2}{x^2+1}}\,dx = 2\pi \int_0^3 \sqrt{2x^2+1}\,dx = 2\sqrt{2}\pi \int_0^3 \sqrt{x^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2}\,dx$$

$$\overset{21}{=} 2\sqrt{2}\pi\left[\tfrac{1}{2}x\sqrt{x^2+\tfrac{1}{2}} + \tfrac{1}{4}\ln\left(x + \sqrt{x^2+\tfrac{1}{2}}\right)\right]_0^3$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{9+\tfrac{1}{2}} + \tfrac{1}{4}\ln\left(3 + \sqrt{9+\tfrac{1}{2}}\right) - \tfrac{1}{4}\ln\tfrac{1}{\sqrt{2}}\right] = 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{\tfrac{19}{2}} + \tfrac{1}{4}\ln\left(3+\sqrt{\tfrac{19}{2}}\right) + \tfrac{1}{4}\ln\sqrt{2}\right]$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\tfrac{\sqrt{19}}{\sqrt{2}} + \tfrac{1}{4}\ln(3\sqrt{2}+\sqrt{19})\right] = 3\sqrt{19}\pi + \tfrac{\pi}{\sqrt{2}}\ln(3\sqrt{2}+\sqrt{19})$$

23. $y = x^3$ and $0 \le y \le 1 \;\Rightarrow\; y' = 3x^2$ and $0 \le x \le 1$.

$$S = \textstyle\int_0^1 2\pi x\sqrt{1 + (3x^2)^2}\,dx = 2\pi \int_0^3 \sqrt{1+u^2}\,\tfrac{1}{6}\,du \qquad [u = 3x^2,\ du = 6x\,dx]$$

$$= \tfrac{\pi}{3}\textstyle\int_0^3 \sqrt{1+u^2}\,du \overset{21}{=} \text{[or use CAS]}\quad \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln(u+\sqrt{1+u^2})\right]_0^3$$

$$= \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10} + \tfrac{1}{2}\ln\left(3+\sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10} + \ln\left(3+\sqrt{10}\right)\right]$$

24. $y = \ln(x+1),\ 0 \le x \le 1.\ ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\dfrac{1}{x+1}\right)^2}\,dx$, so

$$S = \int_0^1 2\pi x\sqrt{1 + \frac{1}{(x+1)^2}}\,dx = \int_1^2 2\pi(u-1)\sqrt{1 + \frac{1}{u^2}}\,du \qquad [u = x+1,\ du = dx]$$

$$= 2\pi\int_1^2 u\,\frac{\sqrt{1+u^2}}{u}\,du - 2\pi\int_1^2 \frac{\sqrt{1+u^2}}{u}\,du = 2\pi\int_1^2 \sqrt{1+u^2}\,du - 2\pi\int_1^2 \frac{\sqrt{1+u^2}}{u}\,du$$

$$\overset{21,\,23}{=} \text{[or use CAS]}\ 2\pi\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_1^2 - 2\pi\left[\sqrt{1+u^2} - \ln\left(\frac{1+\sqrt{1+u^2}}{u}\right)\right]_1^2$$

$$= 2\pi\left[\sqrt{5} + \tfrac{1}{2}\ln\left(2+\sqrt{5}\right) - \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\ln\left(1+\sqrt{2}\right)\right] - 2\pi\left[\sqrt{5} - \ln\left(\tfrac{1+\sqrt{5}}{2}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

$$= 2\pi\left[\tfrac{1}{2}\ln\left(2+\sqrt{5}\right) + \ln\left(\tfrac{1+\sqrt{5}}{2}\right) + \tfrac{\sqrt{2}}{2} - \tfrac{3}{2}\ln\left(1+\sqrt{2}\right)\right]$$

25. $S = 2\pi \int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$$S = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi \int_1^\infty \frac{x^2}{x^3}\,dx = 2\pi \int_1^\infty \frac{1}{x}\,dx$$

But we know that this integral diverges, so the area S is infinite.

26. $S = \int_0^\infty 2\pi y\sqrt{1+(dy/dx)^2}\,dx = 2\pi \int_0^\infty e^{-x}\sqrt{1+(-e^{-x})^2}\,dx \qquad [y = e^{-x},\, y' = -e^{-x}]$.

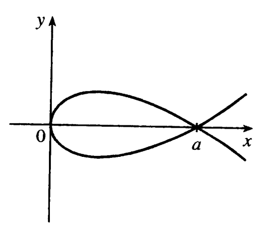
Evaluate $I = \int e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x}\,dx$.

$$\begin{aligned} I &= \int \sqrt{1+u^2}\,du \overset{21}{=} \tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right) + C \\ &= \tfrac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \tfrac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right) + C \end{aligned}$$

Returning to the surface area integral, we have

$$\begin{aligned} S &= 2\pi \lim_{t\to\infty} \int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\,dx \\ &= 2\pi \lim_{t\to\infty} \left[\tfrac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \tfrac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)\right]_0^t \\ &= 2\pi \lim_{t\to\infty} \left\{\left[\tfrac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}} + \tfrac{1}{2}\ln\left(-e^{-t}+\sqrt{1+e^{-2t}}\right)\right] - \left[\tfrac{1}{2}(-1)\sqrt{1+1} + \tfrac{1}{2}\ln\left(-1+\sqrt{1+1}\right)\right]\right\} \\ &= 2\pi\left\{\left[\tfrac{1}{2}(0)\sqrt{1} + \tfrac{1}{2}\ln\left(0+\sqrt{1}\right)\right] - \left[-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(-1+\sqrt{2}\right)\right]\right\} \\ &= 2\pi\left\{[0] + \tfrac{1}{2}\left[\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]\right\} = \pi\left[\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right] \end{aligned}$$

27. Since $a > 0$, the curve $3ay^2 = x(a-x)^2$ only has points with $x \ge 0$. ($3ay^2 \ge 0 \;\Rightarrow\; x(a-x)^2 \ge 0 \;\Rightarrow\; x \ge 0$.) The curve is symmetric about the x-axis (since the equation is unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a, so the curve's loop extends from $x = 0$ to $x = a$.



$$\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a-x)^2] \;\Rightarrow\; 6ay\frac{dy}{dx} = x\cdot 2(a-x)(-1) + (a-x)^2 \;\Rightarrow$$

$$\frac{dy}{dx} = \frac{(a-x)[-2x+a-x]}{6ay} \;\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-x)^2(a-3x)^2}{36a^2}\cdot\frac{3a}{x(a-x)^2}$$

$$\begin{bmatrix}\text{the last fraction}\\ \text{is } 1/y^2\end{bmatrix} = \frac{(a-3x)^2}{12ax} \;\Rightarrow$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{a^2-6ax+9x^2}{12ax}=\frac{12ax}{12ax}+\frac{a^2-6ax+9x^2}{12ax}=\frac{a^2+6ax+9x^2}{12ax}=\frac{(a+3x)^2}{12ax}\text{ for } x\neq 0.$$

(a) $S=\int_{x=0}^{a}2\pi y\,ds=2\pi\int_0^a\frac{\sqrt{x}(a-x)}{\sqrt{3a}}\cdot\frac{a+3x}{\sqrt{12ax}}\,dx=2\pi\int_0^a\frac{(a-x)(a+3x)}{6a}\,dx$

$$=\frac{\pi}{3a}\int_0^a(a^2+2ax-3x^2)\,dx=\frac{\pi}{3a}\left[a^2x+ax^2-x^3\right]_0^a=\frac{\pi}{3a}(a^3+a^3-a^3)=\frac{\pi}{3a}\cdot a^3=\frac{\pi a^2}{3}.$$

Note that we have rotated the top half of the loop about the x-axis. This generates the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$S=2\cdot 2\pi\int_{x=0}^{a}x\,ds=4\pi\int_0^a x\,\frac{a+3x}{\sqrt{12ax}}\,dx=\frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx$$

$$=\frac{2\pi}{\sqrt{3a}}\int_0^a(ax^{1/2}+3x^{3/2})\,dx=\frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2}+\frac{6}{5}x^{5/2}\right]_0^a=\frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2}+\frac{6}{5}a^{5/2}\right)$$

$$=\frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2=\frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2=\frac{56\pi\sqrt{3}\,a^2}{45}$$

28. In general, if the parabola $y=ax^2$, $-c\leq x\leq c$, is rotated about the y-axis, the surface area it generates is

$$2\pi\int_0^c x\sqrt{1+(2ax)^2}\,dx=2\pi\int_0^{2ac}\frac{u}{2a}\sqrt{1+u^2}\,\frac{1}{2a}\,du\quad\left[\begin{array}{l}u=2ax,\\ du=2a\,dx\end{array}\right]$$

$$=\frac{\pi}{4a^2}\int_0^{2ac}(1+u^2)^{1/2}\,2u\,du=\frac{\pi}{4a^2}\left[\tfrac{2}{3}(1+u^2)^{3/2}\right]_0^{2ac}$$

$$=\frac{\pi}{6a^2}\left[(1+4a^2c^2)^{3/2}-1\right]$$

Here $2c=10$ ft and $ac^2=2$ ft, so $c=5$ and $a=\frac{2}{25}$. Thus, the surface area is

$$S=\tfrac{\pi}{6}\tfrac{625}{4}\left[\left(1+4\cdot\tfrac{4}{625}\cdot 25\right)^{3/2}-1\right]=\tfrac{625\pi}{24}\left[\left(1+\tfrac{16}{25}\right)^{3/2}-1\right]=\tfrac{625\pi}{24}\left(\tfrac{41\sqrt{41}}{125}-1\right)$$

$$=\tfrac{5\pi}{24}\left(41\sqrt{41}-125\right)\approx 90.01\text{ ft}^2$$

29. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\quad\Rightarrow\quad\dfrac{y\,(dy/dx)}{b^2}=-\dfrac{x}{a^2}\quad\Rightarrow\quad\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}\quad\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^4x^2}{a^4y^2}=\frac{b^4x^2+a^4y^2}{a^4y^2}=\frac{b^4x^2+a^4b^2(1-x^2/a^2)}{a^4b^2(1-x^2/a^2)}=\frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$=\frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2}=\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x-axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,dx = \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}} \qquad \left[u = \sqrt{a^2-b^2}\,x\right]$$

$$\overset{30}{=} \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2} + \frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2(a^2-b^2)} + \frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right] = 2\pi\left[b^2 + \frac{a^2b\sin^{-1}\dfrac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

30. The upper half of the torus is generated by rotating the curve $(x-R)^2+y^2=r^2$, $y>0$, about the y-axis.

$$y\,\frac{dy}{dx} = -(x-R) \quad\Rightarrow\quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-R)^2}{y^2} = \frac{y^2+(x-R)^2}{y^2} = \frac{r^2}{r^2-(x-R)^2}.\ \text{Thus,}$$

$$S = 2\int_{R-r}^{R+r} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_{R-r}^{R+r}\frac{rx}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$= 4\pi r\int_{-r}^{r}\frac{u+R}{\sqrt{r^2-u^2}}\,du \qquad [u = x-R]$$

$$= 4\pi r\int_{-r}^{r}\frac{u\,du}{\sqrt{r^2-u^2}} + 4\pi Rr\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}$$

$$= 4\pi r\cdot 0 + 8\pi Rr\int_0^r\frac{du}{\sqrt{r^2-u^2}} \qquad \text{[since the first integrand is odd and the second is even]}$$

$$= 8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r = 8\pi Rr\left(\tfrac{\pi}{2}\right) = 4\pi^2 Rr$$

31. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c - f(x_i^*)$, so

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x = \int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$$

32. $y = x^{1/2} \quad\Rightarrow\quad y' = \frac{1}{2}x^{-1/2} \quad\Rightarrow\quad 1+(y')^2 = 1+1/4x$, so by Exercise 31,

$S = \int_0^4 2\pi(4-\sqrt{x})\,\sqrt{1+1/(4x)}\,dx$. Using a CAS, we get

$S = 2\pi\ln(\sqrt{17}+4) + \frac{\pi}{6}(31\sqrt{17}+1) \approx 80.6095.$

33. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

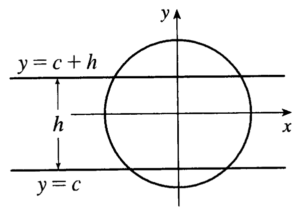
$$S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\pi\int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\,dx$$

$$= 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right)dx$$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \dfrac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right)dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}}\right)dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2\left(\frac{\pi}{2}\right) = 4\pi^2 r^2$.

34. Take the sphere $x^2 + y^2 + z^2 = \frac{1}{4}d^2$ and let the intersecting planes be $y = c$ and $y = c + h$, where $-\frac{1}{2}d \le c \le \frac{1}{2}d - h$. The sphere intersects the xy-plane in the circle $x^2 + y^2 = \frac{1}{4}d^2$. From this equation, we get $x\,\dfrac{dx}{dy} + y = 0$, so $\dfrac{dx}{dy} = -\dfrac{y}{x}$. The desired surface area is



$$S = 2\pi \int x\,ds = 2\pi \int_c^{c+h} x\sqrt{1 + (dx/dy)^2}\,dy = 2\pi \int_c^{c+h} x\sqrt{1 + y^2/x^2}\,dy = 2\pi \int_c^{c+h} \sqrt{x^2 + y^2}\,dy$$

$$= 2\pi \int_c^{c+h} \tfrac{1}{2}d\,dy = \pi d \int_c^{c+h} dy = \pi d h$$

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\,\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ must be replaced by $y_i = |f(x_i)| \approx |f(x_i^*)|$ and $y_{i-1} = |f(x_{i-1})| \approx |f(x_i^*)|$. Thus, $2\pi\,\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i| \approx 2\pi\,|f(x_i^*)|\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\,|f(x)|\sqrt{1 + [f'(x)]^2}\,dx$.

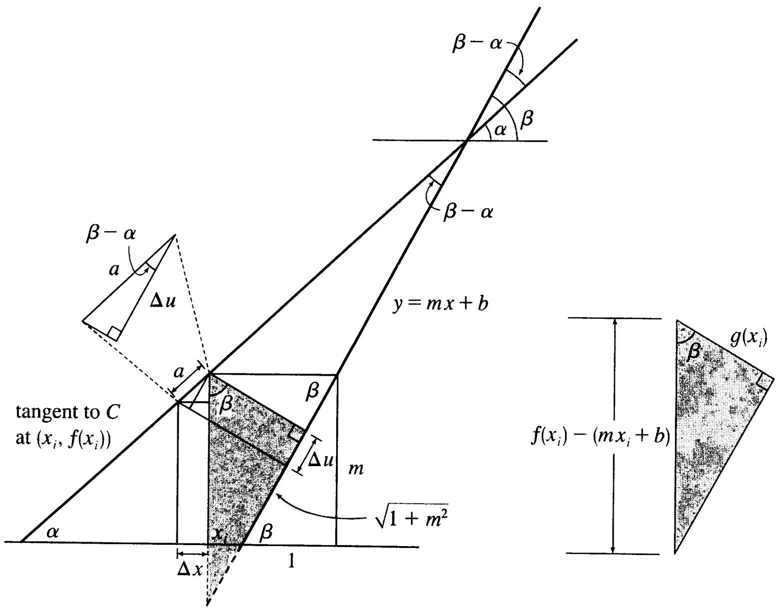
36. Since $g(x) = f(x) + c$, we have $g'(x) = f'(x)$. Thus,

$$S_g = \int_a^b 2\pi g(x)\sqrt{1 + [g'(x)]^2}\,dx = \int_a^b 2\pi[f(x) + c]\sqrt{1 + [f'(x)]^2}\,dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx + 2\pi c\int_a^b \sqrt{1 + [f'(x)]^2}\,dx = S_f + 2\pi c L$$

DISCOVERY PROJECT Rotating on a Slant

1.



In the figure, the segment a lying above the interval $[x_i - \Delta x, x_i]$ along the tangent to C has length $\Delta x \sec\alpha = \Delta x\sqrt{1+\tan^2\alpha} = \sqrt{1+[f'(x_i)]^2}\,\Delta x$. The segment from $(x_i, f(x_i))$ drawn perpendicular to the line $y = mx + b$ has length

$$g(x_i) = [f(x_i) - mx_i - b]\cos\beta = \frac{f(x_i) - mx_i - b}{\sec\beta} = \frac{f(x_i) - mx_i - b}{\sqrt{1+\tan^2\beta}} = \frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}$$

Also, $\cos(\beta-\alpha) = \dfrac{\Delta u}{\Delta x\,\sec\alpha} \quad \Rightarrow$

$$\Delta u = \Delta x\,\sec\alpha\,\cos(\beta-\alpha) = \Delta x\,\frac{\cos\beta\,\cos\alpha + \sin\beta\,\sin\alpha}{\cos\alpha} = \Delta x(\cos\beta + \sin\beta\,\tan\alpha)$$

$$= \Delta x\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x_i)\right] = \frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

Thus,

$$\text{Area}(\mathfrak{R}) = \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\cdot\frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

$$= \frac{1}{1+m^2}\int_p^q [f(x) - mx - b]\,[1 + mf'(x)]\,dx$$

2. From Problem 1 with $m = 1$, $f(x) = x + \sin x$, $mx + b = x - 2$, $p = 0$, and $q = 2\pi$,

$$\begin{aligned}\text{Area} &= \tfrac{1}{1+1^2}\int_0^{2\pi}[x + \sin x - (x-2)]\,[1 + 1(1+\cos x)]\,dx = \tfrac{1}{2}\int_0^{2\pi}(\sin x + 2)(2 + \cos x)\,dx \\ &= \tfrac{1}{2}\int_0^{2\pi}(2\sin x + \sin x\cos x + 4 + 2\cos x)\,dx = \tfrac{1}{2}\left[-2\cos x + \tfrac{1}{2}\sin^2 x + 4x + 2\sin x\right]_0^{2\pi} \\ &= \tfrac{1}{2}[(-2 + 0 + 8\pi + 0) - (-2 + 0 + 0 + 0)] = \tfrac{1}{2}(8\pi) = 4\pi\end{aligned}$$

3. $$\begin{aligned}V &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\,[g(x_i)]^2\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\right]^2\frac{1 + mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x \\ &= \frac{\pi}{(1+m^2)^{3/2}}\int_p^q [f(x) - mx - b]^2\,[1 + mf'(x)]\,dx\end{aligned}$$

4. $$\begin{aligned}V &= \frac{\pi}{(1+1^2)^{3/2}}\int_0^{2\pi}(x + \sin x - x + 2)^2(1 + 1 + \cos x)\,dx \\ &= \frac{\pi}{2\sqrt{2}}\int_0^{2\pi}(\sin x + 2)^2(\cos x + 2)\,dx \\ &= \tfrac{\pi}{2\sqrt{2}}\int_0^{2\pi}(\sin^2 x + 4\sin x + 4)(\cos x + 2)\,dx \\ &= \tfrac{\pi}{2\sqrt{2}}\int_0^{2\pi}(\sin^2 x\,\cos x + 4\sin x\,\cos x + 4\cos x + 2\sin^2 x + 8\sin x + 8)\,dx \\ &= \tfrac{\pi}{2\sqrt{2}}\left[\tfrac{1}{3}\sin^3 x + 2\sin^2 x + 4\sin x + x - \tfrac{1}{2}\sin 2x - 8\cos x + 8x\right]_0^{2\pi} \qquad [\text{since } 2\sin^2 x = 1 - \cos 2x] \\ &= \tfrac{\pi}{2\sqrt{2}}[(2\pi - 8 + 16\pi) - (-8)] = \tfrac{9\sqrt{2}}{2}\pi^2\end{aligned}$$

5. $$S = \int_p^q 2\pi g(x)\sqrt{1 + [f'(x)]^2}\,dx = \frac{2\pi}{\sqrt{1+m^2}}\int_p^q [f(x) - mx - b]\sqrt{1 + [f'(x)]^2}\,dx$$

6. From Problem 5 with $f(x) = \sqrt{x}$, $p = 0$, $q = 4$, $m = \frac{1}{2}$, and $b = 0$,

$$\begin{aligned}S &= \frac{2\pi}{\sqrt{1 + \left(\frac{1}{2}\right)^2}}\int_0^4\left(\sqrt{x} - \tfrac{1}{2}x\right)\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \\ &= \frac{\pi}{\sqrt{5}}\left[\frac{\ln\left(\sqrt{17} + 4\right)}{32} + \frac{37\sqrt{17}}{24} - \frac{1}{3}\right] \qquad [\text{from CAS}] \\ &\approx 8.554\end{aligned}$$

8.3 Applications to Physics and Engineering

1. The weight density of water is $\delta = 62.5\ \text{lb/ft}^3$.

(a) $P = \delta d \approx (62.5\ \text{lb/ft}^3)(3\ \text{ft}) = 187.5\ \text{lb/ft}^2$

(b) $F = PA \approx (187.5\ \text{lb/ft}^2)(5\ \text{ft})(2\ \text{ft}) = 1875\ \text{lb}$. ($A$ is the area of the bottom of the tank.)

(c) As in Example 1, the area of the ith strip is $2\,(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x \cdot 2\,dx \approx (62.5)(2)\int_0^3 x\,dx = 125\left[\tfrac{1}{2}x^2\right]_0^3 = 125\left(\tfrac{9}{2}\right) = 562.5\ \text{lb}$$

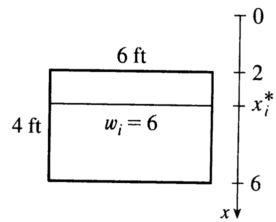
2. (a) $P = \rho g d = 1030(9.8)(2.5) = 25{,}235 \approx 2.52 \times 10^4\ \text{Pa} = 25.2\ \text{kPa}$

(b) $F = PA \approx (2.52 \times 10^4\ \text{N/m}^2)(50\ \text{m}^2) = 1.26 \times 10^6\ \text{N}$

(c) $F = \int_0^{2.5} \rho g x \cdot 5\,dx = (1030)(9.8)(5)\int_0^{2.5} x\,dx \approx 2.52 \times 10^4\left[x^2\right]_0^{2.5} \approx 1.58 \times 10^5\ \text{N}$

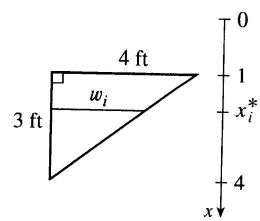
In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

3. Set up a vertical x-axis as shown, with $x = 0$ at the water's surface and x increasing in the downward direction. Then the area of the ith rectangular strip is $6\,\Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5\ \text{lb/ft}^3$). Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6\,\Delta x$ and the total hydrostatic force $\approx \sum_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x$. The total force



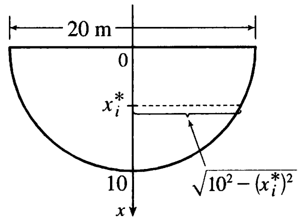
$$F = \lim_{n\to\infty}\sum_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x = \int_2^6 \delta x \cdot 6\,dx = 6\delta\int_2^6 x\,dx$$
$$= 6\delta\left[\tfrac{1}{2}x^2\right]_2^6 = 6\delta(18 - 2) = 96\delta \approx 6000\ \text{lb}$$

4. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is $\frac{4}{3}(4 - x_i^*)\,\Delta x$. [By similar triangles, $\dfrac{w_i}{4 - x_i^*} = \dfrac{4}{3}$, so $w_i = \frac{4}{3}(4 - x_i^*)$.] The pressure on the strip is δx_i^*, so the hydrostatic force on the strip is $\delta x_i^* \cdot \frac{4}{3}(4 - x_i^*)\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \delta x_i^* \cdot \frac{4}{3}(4 - x_i^*)\,\Delta x$. The total force



$$F = \lim_{n\to\infty}\sum_{i=1}^{n} \delta x_i^* \cdot \tfrac{4}{3}(4 - x_i^*)\,\Delta x = \int_1^4 \delta x \cdot \tfrac{4}{3}(4 - x)\,dx = \tfrac{4}{3}\delta\int_1^4 (4x - x^2)\,dx$$
$$= \tfrac{4}{3}\delta\left[2x^2 - \tfrac{1}{3}x^3\right]_1^4 = \tfrac{4}{3}\delta\left[\left(32 - \tfrac{64}{3}\right) - \left(2 - \tfrac{1}{3}\right)\right] = \tfrac{4}{3}\delta(9) = 12\delta \approx 750\ \text{lb}$$

5. Since an equation for the shape is $x^2 + y^2 = 10^2$ $(x \geq 0)$, we have $y = \sqrt{100 - x^2}$. Thus, the area of the ith strip is $2\sqrt{100 - (x_i^*)^2}\,\Delta x$ and the pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x$. The total force

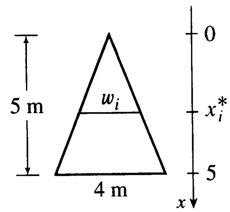


$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100 - (x_i^*)^2}\,\Delta x = \int_0^{10} 2\rho g x\sqrt{100 - x^2}\,dx$$

$$= -\rho g \int_0^{10} \left(100 - x^2\right)^{1/2}(-2x)\,dx = -\rho g\left[\tfrac{2}{3}\left(100 - x^2\right)^{3/2}\right]_0^{10} = -\tfrac{2}{3}\rho g(0 - 1000)$$

$$= \tfrac{2000}{3}\rho g \approx \tfrac{2000}{3} \cdot 1000 \cdot 9.8 \approx 6.5 \times 10^6 \text{ N} \quad \left[\rho \approx 1000 \text{ kg/m}^3 \text{ and } g \approx 9.8 \text{ m/s}^2.\right]$$

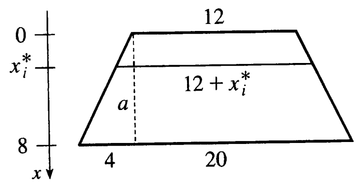
6. By similar triangles, $w_i/4 = x_i^*/5$, so $w_i = \frac{4}{5}x_i^*$ and the area of the ith strip is $\frac{4}{5}x_i^*\,\Delta x$. The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{4}{5}x_i^*\,\Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{4}{5}x_i^*\,\Delta x$. The total force



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot \tfrac{4}{5}x_i^*\,\Delta x = \int_0^5 \rho g x \cdot \tfrac{4}{5}x\,dx = \tfrac{4}{5}\rho g\left[\tfrac{1}{3}x^3\right]_0^5 = \tfrac{4}{5}\rho g \cdot \tfrac{125}{3} = \tfrac{100}{3}\rho g$$

$$\approx \tfrac{100}{3} \cdot 1000 \cdot 9.8 \approx 3.3 \times 10^5 \text{ N.}$$

7. Using similar triangles, $\dfrac{4 \text{ ft wide}}{8 \text{ ft high}} = \dfrac{a \text{ ft wide}}{x_i^* \text{ ft high}}$, so $a = \frac{1}{2}x_i^*$ and the width of the ith rectangular strip is $12 + 2a = 12 + x_i^*$. The area of the strip is $(12 + x_i^*)\,\Delta x$. The pressure on the strip is δx_i^*.

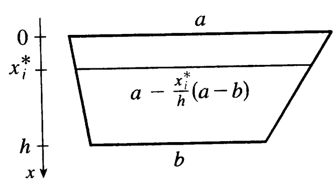


$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^*(12 + x_i^*)\,\Delta x = \int_0^8 \delta x \cdot (12 + x)\,dx$$

$$= \delta \int_0^8 \left(12x + x^2\right)dx = \delta\left[6x^2 + \tfrac{x^3}{3}\right]_0^8 = \delta\left(384 + \tfrac{512}{3}\right)$$

$$= (62.5)\tfrac{1664}{3} \approx 3.47 \times 10^4 \text{ lb}$$

8. In the figure, deleting a $b \times h$ rectangle leaves a triangle with base $a - b$ and height h. By similar triangles, $\dfrac{(a - b) \text{ ft wide}}{h \text{ ft high}} = \dfrac{d \text{ ft wide}}{(h - x_i^*) \text{ ft high}}$, so the width of the triangle is



$$d = \frac{h - x_i^*}{h}(a - b) = \left(1 - \frac{x_i^*}{h}\right)(a - b) = a - b - \frac{x_i^*}{h}(a - b)$$

and the width of the trapezoid is $b + d = a - \dfrac{x_i^*}{h}(a - b)$. The area of the ith rectangular strip is

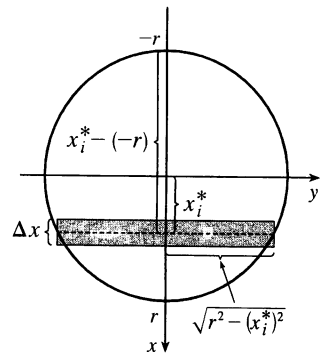
$\left[a - \frac{x_i^*}{h}(a-b)\right] \Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \left[a - \frac{x_i^*}{h}(a-b)\right] \Delta x = \int_0^h \rho g x \left[a - \frac{x}{h}(a-b)\right] dx$$

$$= \rho g a \int_0^h x\,dx + \frac{\rho g(b-a)}{h} \int_0^h x^2\,dx = \rho g a \frac{h^2}{2} + \rho g \frac{b-a}{h}\frac{h^3}{3}$$

$$= \rho g h^2 \left(\frac{a}{2} + \frac{b-a}{3}\right) = \rho g h^2 \frac{a+2b}{6} \approx \frac{500}{3} g h^2 (a+2b)\text{ N}$$

9. From the figure, the area of the ith rectangular strip is $2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$ and the pressure on it is $\rho g(x_i^* + r)$.



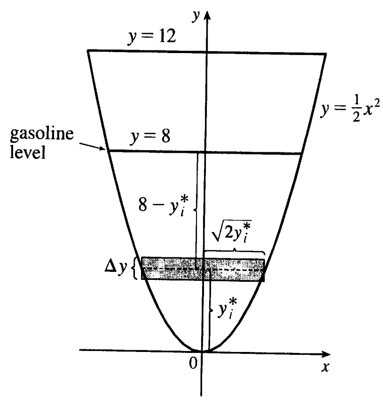
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(x_i^* + r)\, 2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$$

$$= \int_{-r}^{r} \rho g(x+r) \cdot 2\sqrt{r^2 - x^2}\,dx$$

$$= \rho g \int_{-r}^{r} \sqrt{r^2 - x^2}\,2x\,dx + 2\rho g r \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r, or we could make the trigonometric substitution $x = r\sin\theta$. Continuing: $F = \rho g \cdot 0 + 2\rho g r \cdot \frac{1}{2}\pi r^2 = \rho g \pi r^3 = 1000 g \pi r^3$ N (SI units assumed).

10. The area of the ith rectangular strip is $2\sqrt{2y_i^*}\,\Delta y$ and the pressure on it is $\delta d_i = \delta(8 - y_i^*)$.



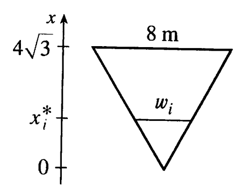
$$F = \int_0^8 \delta(8-y) 2\sqrt{2y}\,dy = 42 \cdot 2 \cdot \sqrt{2} \int_0^8 (8-y) y^{1/2}\,dy$$

$$= 84\sqrt{2} \int_0^8 \left(8y^{1/2} - y^{3/2}\right) dy = 84\sqrt{2}\left[8 \cdot \tfrac{2}{3} y^{3/2} - \tfrac{2}{5} y^{5/2}\right]_0^8$$

$$= 84\sqrt{2}\left[8 \cdot \tfrac{2}{3} \cdot 16\sqrt{2} - \tfrac{2}{5} \cdot 128\sqrt{2}\right]$$

$$= 84\sqrt{2} \cdot 256\sqrt{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 43{,}008 \cdot \tfrac{2}{15} = 5734.4\text{ lb}$$

11. By similar triangles, $\frac{8}{4\sqrt{3}} = \frac{w_i}{x_i^*} \;\Rightarrow\; w_i = \frac{2x_i^*}{\sqrt{3}}$. The area of the ith rectangular strip is $\frac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g\left(4\sqrt{3} - x_i^*\right)$.

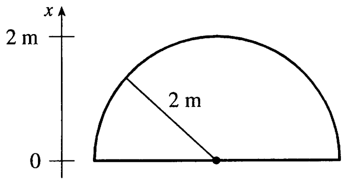


$$F = \int_0^{4\sqrt{3}} \rho g\left(4\sqrt{3} - x\right) \frac{2x}{\sqrt{3}}\,dx = 8\rho g \int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}} \int_0^{4\sqrt{3}} x^2\,dx$$

$$= 4\rho g\left[x^2\right]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}} 64 \cdot 3\sqrt{3}$$

$$= 192\rho g - 128\rho g = 64\rho g \approx 64(840)(9.8) \approx 5.27 \times 10^5\text{ N}$$

12. $F = \int_0^2 \rho g(10 - x)2\sqrt{4 - x^2}\,dx$



$= 20\rho g \int_0^2 \sqrt{4 - x^2}\,dx - \rho g \int_0^2 \sqrt{4 - x^2}\,2x\,dx$

$= 20\rho g \frac{1}{4}\pi(2^2) - \rho g \int_0^4 u^{1/2}\,du \quad [u = 4 - x^2,\ du = -2x\,dx]$

$= 20\pi\rho g - \frac{2}{3}\rho g \left[u^{3/2}\right]_0^4 = 20\pi\rho g - \frac{16}{3}\rho g = \rho g\left(20\pi - \frac{16}{3}\right)$

$= (1000)(9.8)\left(20\pi - \frac{16}{3}\right) \approx 5.63 \times 10^5$ N

13. (a) The top of the cube has depth $d = 1$ m -20 cm $= 80$ cm $= 0.8$ m.

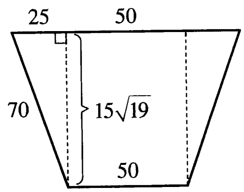
$$F = \rho g d A \approx (1000)(9.8)(0.8)(0.2)^2 = 313.6 \approx 314 \text{ N}$$

(b) The area of a strip is $0.2\,\Delta x$ and the pressure on it is $\rho g x_i^*$.

$F = \int_{0.8}^1 \rho g x(0.2)\,dx = 0.2\rho g\left[\frac{1}{2}x^2\right]_{0.8}^1 = (0.2\rho g)(0.18) = 0.036\rho g = 0.036(1000)(9.8)$

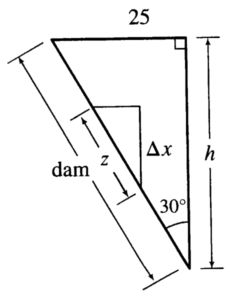
$= 352.8 \approx 353$ N

14. The height of the dam is $h = \sqrt{70^2 - 25^2}\cos 30^\circ = 15\sqrt{19}\left(\frac{\sqrt{3}}{2}\right)$.



From the solution for Exercise 8, the width of the trapezoid is

$100 - \frac{x}{h}(100 - 50) = 100 - \frac{50x}{h}$. From the small triangle in the



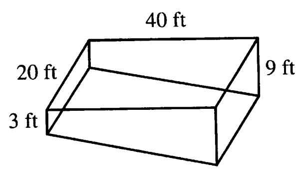
second figure, $\cos 30^\circ = \frac{\Delta x}{z} \quad \Rightarrow \quad z = \Delta x \sec 30^\circ = 2\,\Delta x/\sqrt{3}$.

$$F = \int_0^h \delta x\left(100 - \frac{50x}{h}\right)\frac{2}{\sqrt{3}}\,dx = \frac{200\delta}{\sqrt{3}}\int_0^h x\,dx - \frac{100\delta}{h\sqrt{3}}\int_0^h x^2\,dx$$

$$= \frac{200\delta}{\sqrt{3}}\frac{h^2}{2} - \frac{100\delta}{h\sqrt{3}}\frac{h^3}{3} = \frac{200\delta h^2}{3\sqrt{3}} = \frac{200(62.5)}{3\sqrt{3}} \cdot \frac{12{,}825}{4}$$

$\approx 7.71 \times 10^6$ lb

15. (a) The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i.



$$F = \int_0^3 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^3 = 20\delta \cdot \frac{9}{2} = 90\delta$$

$= 90(62.5) = 5625$ lb $\approx 5.63 \times 10^3$ lb

(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^9 = 20\delta \cdot \frac{81}{2} = 810\delta = 810(62.5) = 50{,}625$ lb $\approx 5.06 \times 10^4$ lb.

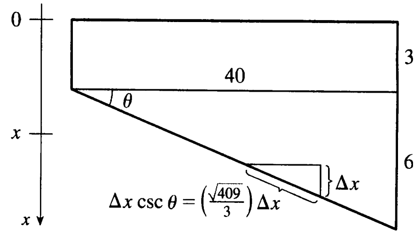
(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a: $\dfrac{a}{40} = \dfrac{9-x}{6} \quad \Rightarrow \quad a = 40 \cdot \dfrac{9-x}{6}$.

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x(40)\tfrac{9-x}{6}\,dx = 40\delta\left[\tfrac{1}{2}x^2\right]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx$$
$$= 180\delta + \tfrac{20}{3}\delta\left[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\right]_3^9 = 180\delta + \tfrac{20}{3}\delta\left[\left(\tfrac{729}{2} - 243\right) - \left(\tfrac{81}{2} - 9\right)\right]$$
$$= 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom,

$$\csc\theta = \frac{\Delta x}{\text{hypotenuse}} \quad \Rightarrow$$

$$\text{hypotenuse} = \Delta x \csc\theta = \Delta x\,\frac{\sqrt{40^2 + 6^2}}{6} = \frac{\sqrt{409}}{3}\,\Delta x.$$



$$F = \int_3^9 \delta x 20\,\tfrac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\left(20\sqrt{409}\right)\delta\left[\tfrac{1}{2}x^2\right]_3^9$$
$$= \tfrac{1}{3} \cdot 10\sqrt{409}\,\delta(81 - 9)$$
$$\approx 303{,}356 \text{ lb} \approx 3.03 \times 10^5 \text{ lb}$$

16. Partition the interval $[a, b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i. The ith horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w(x_i^*)$, so its area is $A_i \approx w(x_i^*)\,\Delta x_i$. For small Δx_i, the pressure P_i on the ith strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the ith strip is $F_i = P_i A_i \approx \rho g x_i^* w(x_i^*)\,\Delta x_i$. Adding these forces and taking the limit as $n \to \infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} F_i = \lim_{n\to\infty}\sum_{i=1}^{n} \rho g x_i^* w(x_i^*)\,\Delta x_i = \int_a^b \rho g x w(x)\,dx$$

17. $F = \int_2^5 \rho g x \cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x. Since $n = 6$, $\Delta x = \frac{5-2}{6} = \frac{1}{2}$, and

$$F \approx S_6 = \rho g \cdot \tfrac{1/2}{3}[2 \cdot w(2) + 4 \cdot 2.5 \cdot w(2.5) + 2 \cdot 3 \cdot w(3) + 4 \cdot 3.5 \cdot w(3.5)$$
$$+ 2 \cdot 4 \cdot w(4) + 4 \cdot 4.5 \cdot w(4.5) + 5 \cdot w(5)]$$
$$= \tfrac{1}{6}\rho g(2 \cdot 0 + 10 \cdot 0.8 + 6 \cdot 1.7 + 14 \cdot 2.4 + 8 \cdot 2.9 + 18 \cdot 3.3 + 5 \cdot 3.6)$$
$$= \tfrac{1}{6}(1000)(9.8)(152.4) \approx 2.5 \times 10^5 \text{ N}$$

18. (a) From Equation 8, $\overline{x} = \dfrac{1}{A}\displaystyle\int_a^b xw(x)\,dx \quad \Rightarrow \quad A\overline{x} = \int_a^b xw(x)\,dx \quad \Rightarrow \quad \rho g A\overline{x} = \rho g\int_a^b xw(x)\,dx \quad \Rightarrow$

$(\rho g\overline{x})A = \int_a^b \rho g\,xw(x)\,dx = F$ by Exercise 16.

(b) The centroid of a circle is its center. In this case, the center is at a depth of r meters, so $\overline{x} = r$. Thus, $F = (\rho g\overline{x})A = (\rho g r)(\pi r^2) = \rho g\pi r^3$.

19. The moment M of the system about the origin is $M = \sum_{i=1}^{2} m_i x_i = m_1 x_1 + m_2 x_2 = 40 \cdot 2 + 30 \cdot 5 = 230$.

The mass m of the system is $m = \sum_{i=1}^{2} m_i = m_1 + m_2 = 40 + 30 = 70$. The center of mass of the system is $M/m = \frac{230}{70} = \frac{23}{7}$.

20. $M = m_1x_1 + m_2x_2 + m_3x_3 = 25(-2) + 20(3) + 10(7) = 80;$

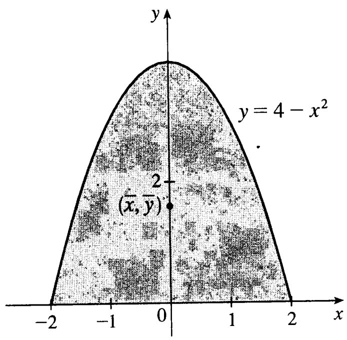
$\overline{x} = M/(m_1 + m_2 + m_3) = \frac{80}{55} = \frac{16}{11}.$

21. $m = \sum_{i=1}^{3} m_i = 6 + 5 + 10 = 21.$ $M_x = \sum_{i=1}^{3} m_i y_i = 6(5) + 5(-2) + 10(-1) = 10;$

$M_y = \sum_{i=1}^{3} m_i x_i = 6(1) + 5(3) + 10(-2) = 1.$ $\overline{x} = \dfrac{M_y}{m} = \dfrac{1}{21}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21}, \frac{10}{21}\right)$.

22. $M_x = \sum_{i=1}^{4} m_i y_i = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3$, $M_y = \sum_{i=1}^{4} m_i x_i = 6(1) + 5(3) + 1(-3) + 4(6) = 42$, and $m = \sum_{i=1}^{4} m_i = 16$, so $\overline{x} = \dfrac{M_y}{m} = \dfrac{42}{16} = \dfrac{21}{8}$ and $\overline{y} = \dfrac{M_x}{m} = -\dfrac{3}{16}$; the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{21}{8}, -\frac{3}{16}\right)$.

23. Since the region in the figure is symmetric about the y-axis, we know that $\overline{x} = 0$. The region is "bottom-heavy," so we know that $\overline{y} < 2$, and we might guess that $\overline{y} = 1.5$.



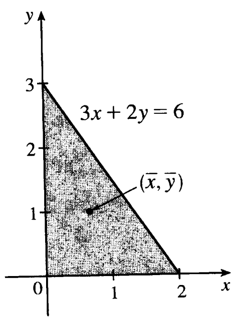
$A = \int_{-2}^{2} (4 - x^2)\,dx = 2\int_0^2 (4 - x^2)\,dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2$
$= 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$

$\overline{x} = \dfrac{1}{A}\displaystyle\int_{-2}^{2} x(4 - x^2)\,dx = 0$ since $f(x) = x(4 - x^2)$ is an odd function (or since the region is symmetric about the y-axis).

$$\overline{y} = \frac{1}{A}\int_{-2}^{2} \frac{1}{2}(4 - x^2)^2\,dx = \frac{3}{32}\cdot\frac{1}{2}\cdot 2\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$
$$= \tfrac{3}{32}\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 3\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 3\left(\tfrac{8}{15}\right) = \tfrac{8}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

24. The region in the figure is "left-heavy" and "bottom-heavy," so we know $\overline{x} < 1$ and $\overline{y} < 1.5$, and we might guess that $\overline{x} = 0.7$ and $\overline{y} = 1.2$.



$3x + 2y = 6 \quad\Leftrightarrow\quad 2y = 6 - 3x \quad\Leftrightarrow\quad y = 3 - \frac{3}{2}x.$

$A = \int_0^2 \left(3 - \frac{3}{2}x\right)dx = \left[3x - \frac{3}{4}x^2\right]_0^2 = 6 - 3 = 3.$

$\overline{x} = \frac{1}{A}\int_0^2 x\left(3 - \frac{3}{2}x\right)dx = \frac{1}{3}\int_0^2 \left(3x - \frac{3}{2}x^2\right)dx$
$= \frac{1}{3}\left[\frac{3}{2}x^2 - \frac{1}{2}x^3\right]_0^2 = \frac{1}{3}(6 - 4) = \frac{2}{3};$

$\overline{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\left(3 - \frac{3}{2}x\right)^2 dx = \frac{1}{3}\cdot\frac{1}{2}\int_0^2 \left(9 - 9x + \frac{9}{4}x^2\right)dx = \frac{1}{6}\left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3\right]_0^2 = \frac{1}{6}(18 - 18 + 6) = 1.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{2}{3}, 1\right)$.

25. The region in the figure is "right-heavy" and "bottom-heavy," so we know $\overline{x} > 0.5$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 0.6$ and $\overline{y} = 0.9$.

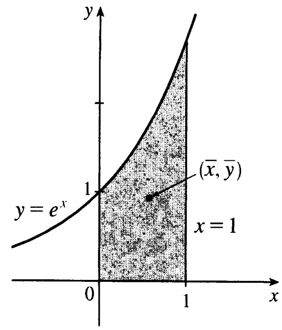
$A = \int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1,$

$\overline{x} = \frac{1}{A}\int_0^1 xe^x\,dx = \frac{1}{e-1}\left[xe^x - e^x\right]_0^1$ [by parts]

$= \frac{1}{e-1}[0 - (-1)] = \frac{1}{e-1},$

$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(e^x)^2\,dx = \frac{1}{e-1}\cdot\frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e-1)}\left(e^2 - 1\right) = \frac{e+1}{4}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.



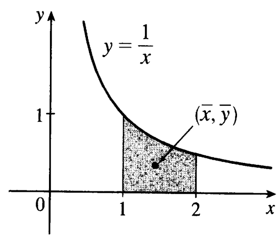
26. The region in the figure is "left-heavy" and "bottom-heavy," so we know $\overline{x} < 1.5$ and $\overline{y} < 0.5$, and we might guess that $\overline{x} = 1.4$ and $\overline{y} = 0.4$.

$A = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2,\ \overline{x} = \frac{1}{A}\int_1^2 x\cdot\frac{1}{x}\,dx = \frac{1}{A}[x]_1^2 = \frac{1}{A} = \frac{1}{\ln 2},$

$\overline{y} = \frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx = \frac{1}{2A}\int_1^2 x^{-2}\,dx = \frac{1}{2A}\left[-\frac{1}{x}\right]_1^2$

$= \frac{1}{2\ln 2}\left(-\frac{1}{2} + 1\right) = \frac{1}{4\ln 2}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right) \approx (1.44, 0.36)$.



27. $A = \int_0^1 (\sqrt{x} - x)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$

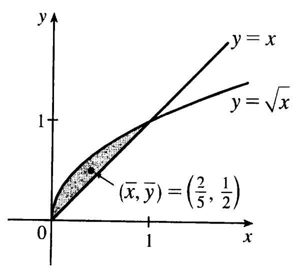
$\overline{x} = \frac{1}{A}\int_0^1 x(\sqrt{x} - x)\,dx = 6\int_0^1 (x^{3/2} - x^2)\,dx$

$= 6\left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 = 6\left(\frac{2}{5} - \frac{1}{3}\right) = 6\left(\frac{1}{15}\right) = \frac{2}{5};$

$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left[(\sqrt{x})^2 - x^2\right] dx = 6\cdot\frac{1}{2}\int_0^1 (x - x^2)\,dx$

$= 3\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = 3\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{2}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{2}{5}, \frac{1}{2}\right)$.



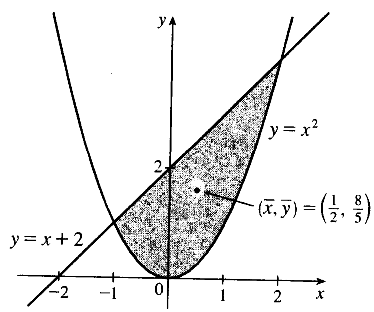
28. $A = \int_{-1}^2 (x + 2 - x^2)\,dx = \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^2$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}.$

$\overline{x} = \frac{1}{A}\int_{-1}^2 x(x + 2 - x^2)\,dx = \frac{2}{9}\int_{-1}^2 \left(x^2 + 2x - x^3\right) dx$

$= \frac{2}{9}\left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right]_{-1}^2$

$= \frac{2}{9}\left[\left(\frac{8}{3} + 4 - 4\right) - \left(-\frac{1}{3} + 1 - \frac{1}{4}\right)\right] = \frac{2}{9}\cdot\frac{9}{4} = \frac{1}{2};$

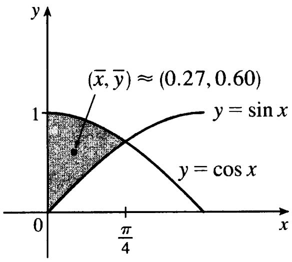


$\overline{y} = \frac{1}{A}\int_{-1}^2 \frac{1}{2}\left[(x+2)^2 - (x^2)^2\right] dx = \frac{2}{9}\cdot\frac{1}{2}\int_{-1}^2 \left(x^2 + 4x + 4 - x^4\right) dx = \frac{1}{9}\left[\frac{1}{3}x^3 + 2x^2 + 4x - \frac{1}{5}x^5\right]_{-1}^2$

$= \frac{1}{9}\left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5}\right) - \left(-\frac{1}{3} + 2 - 4 + \frac{1}{5}\right)\right] = \frac{1}{9}\left(18 + \frac{9}{3} - \frac{33}{5}\right) = \frac{1}{9}\cdot\frac{72}{5} = \frac{8}{5}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{2}, \frac{8}{5}\right)$.

29. $A = \int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$,



$$\overline{x} = A^{-1} \int_0^{\pi/4} x(\cos x - \sin x)\,dx$$

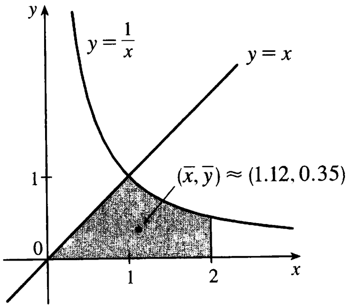
$$= A^{-1}[x(\sin x + \cos x) + \cos x - \sin x]_0^{\pi/4} \quad \text{[integration by parts]}$$

$$= A^{-1}\left(\tfrac{\pi}{4}\sqrt{2} - 1\right) = \frac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1}$$

$$\overline{y} = A^{-1} \int_0^{\pi/4} \tfrac{1}{2}\left(\cos^2 x - \sin^2 x\right) dx = \tfrac{1}{2A} \int_0^{\pi/4} \cos 2x\,dx = \tfrac{1}{4A}[\sin 2x]_0^{\pi/4} = \tfrac{1}{4A} = \tfrac{1}{4(\sqrt{2}-1)}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{\pi\sqrt{2} - 4}{4(\sqrt{2} - 1)}, \dfrac{1}{4(\sqrt{2} - 1)}\right) \approx (0.27, 0.60)$.

30. $A = \int_0^1 x\,dx + \int_1^2 \frac{1}{x}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + [\ln x]_1^2 = \frac{1}{2} + \ln 2$,



$$\overline{x} = \frac{1}{A}\left[\int_0^1 x^2\,dx + \int_1^2 1\,dx\right] = \frac{1}{A}\left(\left[\tfrac{1}{3}x^3\right]_0^1 + [x]_1^2\right)$$

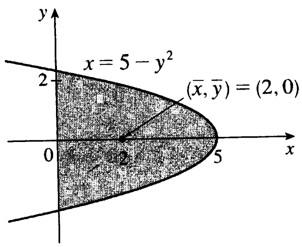
$$= \frac{1}{A}\left(\frac{1}{3} + 1\right) = \frac{2}{1 + 2\ln 2} \cdot \frac{4}{3} = \frac{8}{3(1 + 2\ln 2)},$$

$$\overline{y} = \frac{1}{A}\left[\int_0^1 \tfrac{1}{2}x^2\,dx + \int_1^2 \frac{1}{2x^2}\,dx\right] = \frac{1}{2A}\left(\left[\tfrac{1}{3}x^3\right]_0^1 + \left[-\frac{1}{x}\right]_1^2\right)$$

$$= \frac{1}{2A}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12A} = \frac{5}{6 + 12\ln 2}.$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{8}{3(1 + 2\ln 2)}, \dfrac{5}{6(1 + 2\ln 2)}\right) \approx (1.12, 0.35)$. The principle used in this problem is stated after Example 3: the moment of the union of two nonoverlapping regions is the sum of the moments of the individual regions.

31. From the figure we see that $\overline{y} = 0$. Now



$$A = \int_0^5 2\sqrt{5 - x}\,dx = 2\left[-\tfrac{2}{3}(5 - x)^{3/2}\right]_0^5$$

$$= 2\left(0 + \tfrac{2}{3} \cdot 5^{3/2}\right) = \tfrac{20}{3}\sqrt{5}$$

so

$$\overline{x} = \tfrac{1}{A}\int_0^5 x\left[\sqrt{5 - x} - \left(-\sqrt{5 - x}\right)\right] dx = \tfrac{1}{A}\int_0^5 2x\sqrt{5 - x}\,dx$$

$$= \tfrac{1}{A}\int_{\sqrt{5}}^0 2(5 - u^2)u(-2u)\,du \qquad [u = \sqrt{5 - x},\ x = 5 - u^2,\ u^2 = 5 - x,\ dx = -2u\,du]$$

$$= \tfrac{4}{A}\int_0^{\sqrt{5}} u^2(5 - u^2)\,du = \tfrac{4}{A}\left[\tfrac{5}{3}u^3 - \tfrac{1}{5}u^5\right]_0^{\sqrt{5}} = \tfrac{3}{5\sqrt{5}}\left(\tfrac{25}{3}\sqrt{5} - 5\sqrt{5}\right) = 5 - 3 = 2$$

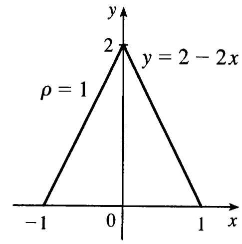
Thus, the centroid is $(\overline{x}, \overline{y}) = (2, 0)$.

32. By symmetry, $M_y = 0$ and $\overline{x} = 0$; $A = \frac{1}{2}\pi \cdot 1^2 + 4$, so $m = \rho A = 5(\frac{\pi}{2} + 4) = \frac{5}{2}(\pi + 8)$;

$M_x = \rho \cdot 2\int_0^1 \frac{1}{2}[(\sqrt{1-x^2}\,)^2 - (-2)^2]dx = 5\int_0^1 (-x^2 - 3)\,dx = -5\left[\frac{1}{3}x^3 + 3x\right]_0^1 = -5 \cdot \frac{10}{3} = -\frac{50}{3}$;

$\overline{y} = \frac{1}{m}M_x = \dfrac{2}{5(\pi+8)} \cdot \frac{-50}{3} = -\dfrac{20}{3(\pi+8)}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \dfrac{-20}{3(\pi+8)}\right)$.

33. By symmetry, $M_y = 0$ and $\overline{x} = 0$. $A = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$.



$$\begin{aligned} M_x &= \rho \int_{-1}^1 \tfrac{1}{2}(2-2x)^2\,dx = 2\rho \int_0^1 \tfrac{1}{2}(2-2x)^2\,dx \\ &= \left(2 \cdot 1 \cdot \tfrac{1}{2} \cdot 2^2\right) \int_0^1 (1-x)^2\,dx \\ &= 4\int_1^0 u^2(-du) \qquad [u = 1-x,\ du = -dx] \\ &= -4\left[\tfrac{1}{3}u^3\right]_1^0 = -4\left(-\tfrac{1}{3}\right) = \tfrac{4}{3} \end{aligned}$$

$\overline{y} = \frac{1}{m}M_x = \frac{1}{\rho A}M_x = \frac{1}{1 \cdot 2} \cdot \frac{4}{3} = \frac{2}{3}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{2}{3}\right)$.

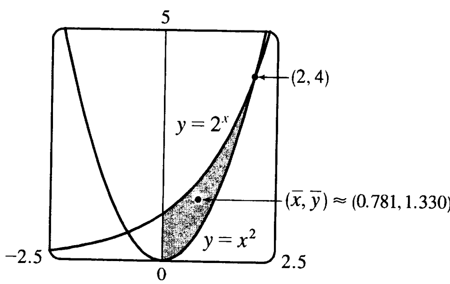
34. By symmetry about the line $y = x$, we expect that $\overline{x} = \overline{y}$. $A = \frac{1}{4}\pi r^2$, so $m = \rho A = 2A = \frac{1}{2}\pi r^2$.

$M_x = \rho \int_0^r \frac{1}{2}\left(\sqrt{r^2 - x^2}\,\right)^2 dx = 2 \cdot \frac{1}{2}\int_0^r (r^2 - x^2)\,dx = \left[r^2x - \frac{1}{3}x^3\right]_0^r = \frac{2}{3}r^3$.

$M_y = \rho \int_0^r x\sqrt{r^2 - x^2}\,dx = \int_0^r (r^2 - x^2)^{1/2} 2x\,dx = \int_0^{r^2} u^{1/2}\,du \;\; [u = r^2 - x^2] = \left[\frac{2}{3}u^{3/2}\right]_0^{r^2} = \frac{2}{3}r^3$.

$\overline{x} = \dfrac{1}{m}M_y = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$; $\overline{y} = \dfrac{1}{m}M_x = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{4}{3\pi}r, \frac{4}{3\pi}r\right)$.

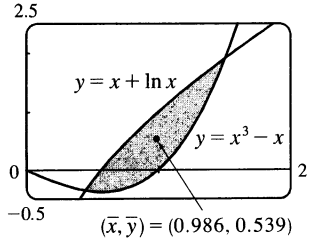
35. $$\begin{aligned} A &= \int_0^2 (2^x - x^2)\,dx = \left[\frac{2^x}{\ln 2} - \frac{x^3}{3}\right]_0^2 \\ &= \left(\frac{4}{\ln 2} - \frac{8}{3}\right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{8}{3} \approx 1.661418. \end{aligned}$$



$$\begin{aligned} \overline{x} &= \frac{1}{A}\int_0^2 x(2^x - x^2)\,dx = \frac{1}{A}\int_0^2 (x2^x - x^3)\,dx \\ &= \frac{1}{A}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - \frac{x^4}{4}\right]_0^2 \qquad \text{[use parts]} \\ &= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} - 4 + \frac{1}{(\ln 2)^2}\right] \\ &= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2} - 4\right] \approx \frac{1}{A}(1.297453) \approx 0.781 \end{aligned}$$

$$\begin{aligned} \overline{y} &= \frac{1}{A}\int_0^2 \tfrac{1}{2}\left[(2^x)^2 - (x^2)^2\right]dx = \frac{1}{A}\int_0^2 \tfrac{1}{2}(2^{2x} - x^4)dx = \frac{1}{A} \cdot \frac{1}{2}\left[\frac{2^{2x}}{2\ln 2} - \frac{x^5}{5}\right]_0^2 \\ &= \frac{1}{A} \cdot \frac{1}{2}\left(\frac{16}{2\ln 2} - \frac{32}{5} - \frac{1}{2\ln 2}\right) = \frac{1}{A}\left(\frac{15}{4\ln 2} - \frac{16}{5}\right) \approx \frac{1}{A}(2.210106) \approx 1.330 \end{aligned}$$

Since the position of a centroid is independent of density when the density is constant, we will assume for convenience that $\rho = 1$ in Exercises 36 and 37.



36. The curves $y = x + \ln x$ and $y = x^3 - x$ intersect at

$(a, c) \approx (0.447141, -0.357742)$ and

$(b, d) \approx (1.507397, 1.917782)$.

$$A = \int_a^b (x + \ln x - x^3 + x)\,dx = \int_a^b (2x + \ln x - x^3)\,dx$$

$$\overset{100}{=} \left[x^2 + x\ln x - x - \tfrac{1}{4}x^4\right]_a^b \approx 0.709781$$

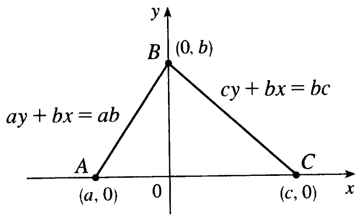
$$\overline{x} = \tfrac{1}{A}\int_a^b x\,(2x + \ln x - x^3)\,dx = \tfrac{1}{A}\int_a^b (2x^2 + x\ln x - x^4)\,dx$$

$$\overset{101}{=} \tfrac{1}{A}\left[\tfrac{2}{3}x^3 + \tfrac{1}{4}x^2(2\ln x - 1) - \tfrac{1}{5}x^5\right]_a^b \approx \tfrac{1}{A}(0.699489) \approx 0.985501$$

$$\overline{y} = \tfrac{1}{A}\int_a^b \tfrac{1}{2}\left[(x + \ln x)^2 - (x^3 - x)^2\right]dx = \tfrac{1}{2A}\int_a^b \left[2x\ln x + (\ln x)^2 - x^6 + 2x^4\right]dx$$

$$\overset{\text{101 and parts}}{=} \tfrac{1}{2A}\left[x^2\ln x - \tfrac{1}{2}x^2 + x(\ln x)^2 - 2x\ln x + 2x - \tfrac{1}{7}x^7 + \tfrac{2}{5}x^5\right]_a^b \approx \tfrac{1}{2A}(0.765092) \approx 0.538964$$

Thus, the centroid is $(\overline{x}, \overline{y}) \approx (0.986, 0.539)$.



37. Choose x- and y-axes so that the base (one side of the triangle) lies along the x-axis with the other vertex along the positive y-axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a + c), 0\right)$ of side AC, so the point of intersection of the medians is $\left(\frac{2}{3} \cdot \frac{1}{2}(a + c), \frac{1}{3}b\right) = \left(\frac{1}{3}(a + c), \frac{1}{3}b\right)$.

This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c - a)b$.

$$\overline{x} = \tfrac{1}{A}\left[\int_a^0 x \cdot \tfrac{b}{a}(a - x)\,dx + \int_0^c x \cdot \tfrac{b}{c}(c - x)\,dx\right] = \tfrac{1}{A}\left[\tfrac{b}{a}\int_a^0 (ax - x^2)\,dx + \tfrac{b}{c}\int_0^c (cx - x^2)\,dx\right]$$

$$= \tfrac{b}{Aa}\left[\tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3\right]_a^0 + \tfrac{b}{Ac}\left[\tfrac{1}{2}cx^2 - \tfrac{1}{3}x^3\right]_0^c = \tfrac{b}{Aa}\left[-\tfrac{1}{2}a^3 + \tfrac{1}{3}a^3\right] + \tfrac{b}{Ac}\left[\tfrac{1}{2}c^3 - \tfrac{1}{3}c^3\right]$$

$$= \tfrac{2}{a(c - a)} \cdot \tfrac{-a^3}{6} + \tfrac{2}{c(c - a)} \cdot \tfrac{c^3}{6} = \tfrac{1}{3(c - a)}(c^2 - a^2) = \tfrac{a + c}{3}$$

and

$$\overline{y} = \tfrac{1}{A}\left[\int_a^0 \tfrac{1}{2}\left(\tfrac{b}{a}(a - x)\right)^2 dx + \int_0^c \tfrac{1}{2}\left(\tfrac{b}{c}(c - x)\right)^2 dx\right]$$

$$= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\int_a^0 (a^2 - 2ax + x^2)\,dx + \tfrac{b^2}{2c^2}\int_0^c (c^2 - 2cx + x^2)\,dx\right]$$

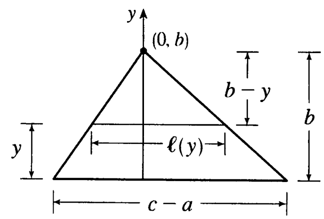
$$= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\left[a^2x - ax^2 + \tfrac{1}{3}x^3\right]_a^0 + \tfrac{b^2}{2c^2}\left[c^2x - cx^2 + \tfrac{1}{3}x^3\right]_0^c\right]$$

$$= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\left(-a^3 + a^3 - \tfrac{1}{3}a^3\right) + \tfrac{b^2}{2c^2}\left(c^3 - c^3 + \tfrac{1}{3}c^3\right)\right] = \tfrac{1}{A}\left[\tfrac{b^2}{6}(-a + c)\right] = \tfrac{2}{(c - a)b} \cdot \tfrac{(c - a)b^2}{6} = \tfrac{b}{3}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{a+c}{3}, \dfrac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\overline{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\overline{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\,\Delta y$, its mass is $\rho\ell(y)\,\Delta y$, and its moment about the x-axis is $\Delta M_x = \rho y \ell(y)\,\Delta y$. Thus,



$$M_x = \int \rho y \ell(y)\,dy \quad \text{and} \quad \overline{y} = \frac{\int \rho y \ell(y)\,dy}{\rho A} = \tfrac{1}{A}\int y\ell(y)\,dy$$

In this problem, $\ell(y) = \dfrac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y} = \frac{1}{A}\int_0^b \frac{c-a}{b}\,y(b-y)dy = \frac{2}{b^2}\int_0^b \left(by - y^2\right)dy = \frac{2}{b^2}\left[\tfrac{1}{2}by^2 - \tfrac{1}{3}y^3\right]_0^b = \frac{2}{b^2}\cdot\frac{b^3}{6} = \frac{b}{3}$$

Notice that only one integral is needed when this method is used.

38. Divide the lamina into three rectangles with masses 2, 2 and 6, with centroids $\left(-\frac{3}{2}, 1\right)$, $\left(0, \frac{1}{2}\right)$ and $\left(2, \frac{3}{2}\right)$, respectively. The total mass of the lamina is 10. So, using Formulas 5, 6, and 7, we have

$\overline{x} = \dfrac{M_y}{m} = \dfrac{1}{m}\sum\limits_{i=1}^{3} m_i x_i = \frac{1}{10}\left[2\left(-\frac{3}{2}\right) + 2(0) + 6(2)\right] = \frac{1}{10}(9)$, and

$\overline{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\sum\limits_{i=1}^{3} m_i y_i = \frac{1}{10}\left[2(1) + 2\left(\frac{1}{2}\right) + 6\left(\frac{3}{2}\right)\right] = \frac{1}{10}(12)$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{10}, \frac{6}{5}\right)$.

39. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 37, the triangles have centroids $\left(-1, \frac{2}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0, -\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have

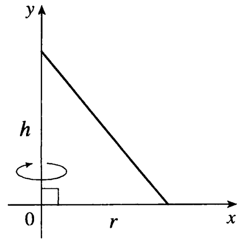
$\overline{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\sum\limits_{i=1}^{3} m_i y_i = \frac{1}{8}\left[2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) + 4\left(-\frac{1}{2}\right)\right] = \frac{1}{8}\left(\frac{2}{3}\right) = \frac{1}{12}$, and $\overline{x} = 0$, since the lamina is symmetric about the line $x = 0$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{1}{12}\right)$.

40. A sphere can be generated by rotating a semicircle about its diameter. By Example 4, the center of mass travels a distance $2\pi\overline{y} = 2\pi\left(\frac{4r}{3\pi}\right) = \frac{8r}{3}$, so by the Theorem of Pappus, the volume of the sphere is

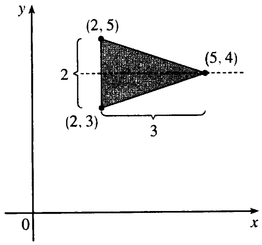
$V = Ad = \dfrac{\pi r^2}{2}\cdot\dfrac{8r}{3} = \frac{4}{3}\pi r^3$.

41. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 37, $\overline{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is

$V = Ad = \left(\frac{1}{2} \cdot \text{base} \cdot \text{height}\right) \cdot (2\pi\overline{x}) = \frac{1}{2}rh \cdot 2\pi\left(\frac{1}{3}r\right) = \frac{1}{3}\pi r^2 h.$



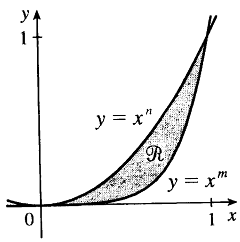
42. From the symmetry in the figure, $\overline{y} = 4$. So the distance traveled by the centroid when rotating the triangle about the x-axis is $d = 2\pi \cdot 4 = 8\pi$. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad = 3(8\pi) = 24\pi$.



43. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the ith subinterval; that is, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$. Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$. Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and $M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty} \sum_i \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and $M_x = \lim\limits_{n\to\infty} \sum_i \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$. Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$$

44. (a) Let $0 \leq x \leq 1$. If $n < m$, then $x^n > x^m$; that is, raising x to a larger power produces a smaller number.



(b) Using Formulas 9 and the fact that the area of $\mathfrak{R}$ is

$A = \displaystyle\int_0^1 (x^n - x^m)\,dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$, we get

$$\overline{x} = \frac{(n+1)(m+1)}{m-n}\int_0^1 x[x^n - x^m]\,dx = \frac{(n+1)(m+1)}{m-n}\int_0^1 \left(x^{n+1} - x^{m+1}\right)dx$$

$$= \frac{(n+1)(m+1)}{m-n}\left[\frac{1}{n+2} - \frac{1}{m+2}\right] = \frac{(n+1)(m+1)}{(n+2)(m+2)}$$

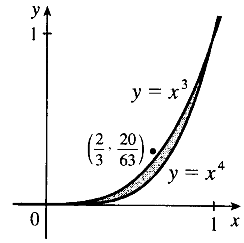
and

$$\overline{y} = \frac{(n+1)(m+1)}{m-n} \int_0^1 \frac{1}{2}\left[(x^n)^2 - (x^m)^2\right] dx = \frac{(n+1)(m+1)}{2(m-n)} \int_0^1 \left(x^{2n} - x^{2m}\right) dx$$
$$= \frac{(n+1)(m+1)}{2(m-n)}\left[\frac{1}{2n+1} - \frac{1}{2m+1}\right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

(c) If we take $n = 3$ and $m = 4$, then

$$(\overline{x}, \overline{y}) = \left(\frac{4 \cdot 5}{5 \cdot 6}, \frac{4 \cdot 5}{7 \cdot 9}\right) = \left(\frac{2}{3}, \frac{20}{63}\right)$$

which lies outside $\mathcal{R}$ since $\left(\frac{2}{3}\right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



8.4 Applications to Economics and Biology

1. By the Net Change Theorem, $C(2000) - C(0) = \int_0^{2000} C'(x)\,dx \quad \Rightarrow$

$$C(2000) = 20{,}000 + \int_0^{2000} (5 - 0.008x + 0.000009x^2)\,dx = 20{,}000 + \left[5x - 0.004x^2 + 0.000003x^3\right]_0^{2000}$$
$$= 20{,}000 + 10{,}000 - 0.004(4{,}000{,}000) + 0.000003(8{,}000{,}000{,}000) = 30{,}000 - 16{,}000 + 24{,}000$$
$$= \$38{,}000$$

2. By the Net Change Theorem, $R(5000) - R(1000) = \int_{1000}^{5000} R'(x)\,dx \quad \Rightarrow$

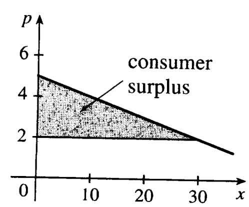
$$R(5000) = 12{,}400 + \int_{1000}^{5000} (12 - 0.0004x)\,dx = 12{,}400 + \left[12x - 0.0002x^2\right]_{1000}^{5000}$$
$$= 12{,}400 + (60{,}000 - 5{,}000) - (12{,}000 - 200) = \$55{,}600$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

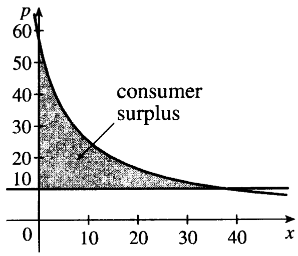
$$C(1600) - C(1200) = \int_{1200}^{1600} C'(x)\,dx = \int_{1200}^{1600} \left(74 + 1.1x - 0.002x^2 + 0.00004x^3\right) dx$$
$$= \left[74x + 0.55x^2 - \tfrac{0.002}{3}x^3 + 0.00001x^4\right]_{1200}^{1600}$$
$$= 64{,}331{,}733.33 - 20{,}464{,}800 = \$43{,}866{,}933.33$$

4. Consumer surplus $= \int_0^{30} [p(x) - p(30)]\,dx$

$$= \int_0^{30} \left[5 - \tfrac{1}{10}x - \left(5 - \tfrac{30}{10}\right)\right] dx$$
$$= \left[3x - \tfrac{1}{20}x^2\right]_0^{30} = 90 - 45 = \$45$$



5. $p(x) = 10 \quad \Rightarrow \quad \dfrac{450}{x+8} = 10 \quad \Rightarrow \quad x + 8 = 45 \quad \Rightarrow \quad x = 37.$



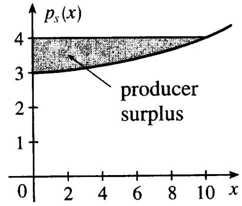
$$\text{Consumer surplus} = \int_0^{37} [p(x) - 10]\, dx = \int_0^{37} \left(\frac{450}{x+8} - 10 \right) dx$$

$$= \left[450 \ln (x+8) - 10x\right]_0^{37}$$

$$= (450 \ln 45 - 370) - 450 \ln 8$$

$$= 450 \ln\left(\tfrac{45}{8}\right) - 370 \approx \$407.25$$

6. $p_S(x) = 3 + 0.01x^2$. $P = p_S(10) = 3 + 1 = 4$.



$$\text{Producer surplus} = \textstyle\int_0^{10} [P - p_S(x)]\, dx$$

$$= \textstyle\int_0^{10} \left[4 - 3 - 0.01x^2\right] dx = \left[x - \frac{0.01}{3} x^3\right]_0^{10}$$

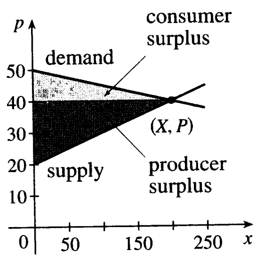
$$\approx 10 - 3.33 = \$6.67$$

7. $P = p_S(x) \quad \Rightarrow \quad 400 = 200 + 0.2x^{3/2} \quad \Rightarrow \quad 200 = 0.2x^{3/2} \quad \Rightarrow \quad 1000 = x^{3/2} \quad \Rightarrow \quad x = 1000^{2/3} = 100.$

$$\text{Producer surplus} = \textstyle\int_0^{100} [P - p_S(x)]\, dx = \int_0^{100} \left[400 - \left(200 + 0.2x^{3/2}\right)\right] dx = \int_0^{100} \left(200 - \frac{1}{5} x^{3/2}\right) dx$$

$$= \left[200x - \tfrac{2}{25} x^{5/2}\right]_0^{100} = 20{,}000 - 8{,}000 = \$12{,}000$$

8. $p = 50 - \frac{1}{20}x$ and $p = 20 + \frac{1}{10}x$ intersect at $p = 40$ and $x = 200$.



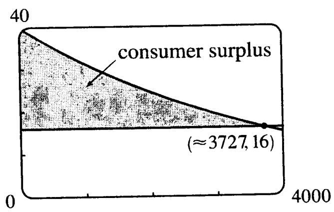
$$\text{Consumer surplus} = \textstyle\int_0^{200} \left(50 - \frac{1}{20}x - 40\right) dx$$

$$= \left[10x - \tfrac{1}{40}x^2\right]_0^{200} = \$1000$$

$$\text{Producer surplus} = \textstyle\int_0^{200} \left(40 - 20 - \frac{1}{10}x\right) dx$$

$$= \left[20x - \tfrac{1}{20}x^2\right]_0^{200} = \$2000$$

9. $p(x) = \dfrac{800{,}000 e^{-x/5000}}{x + 20{,}000} = 16 \quad \Rightarrow \quad x = x_1 \approx 3727.04.$



$$\text{Consumer surplus} = \textstyle\int_0^{x_1} [p(x) - 16]\, dx \approx \$37{,}753$$

10. The demand function is linear with slope $\frac{-0.5}{35} = -\frac{1}{70}$ and $p(400) = 7.5$, so an equation is $p - 7.5 = -\frac{1}{70}(x - 400)$ or $p = -\frac{1}{70}x + \frac{185}{14}$. A selling price of \$6 implies that $6 = -\frac{1}{70}x + \frac{185}{14} \Rightarrow$ $\frac{1}{70}x = \frac{185}{14} - \frac{84}{14} = \frac{101}{14} \quad \Rightarrow \quad x = 505.$

$$\text{Consumer surplus} = \textstyle\int_0^{505} \left(-\frac{1}{70}x + \frac{185}{14} - 6\right) dx = \left[-\frac{1}{140}x^2 + \frac{101}{14}x\right]_0^{505} \approx \$1821.61$$

11. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^8 = \frac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

12. $$n(9) - n(5) = \int_5^9 \left(2200 + 10e^{0.8t}\right) dt = \left[2200t + \frac{10e^{0.8t}}{0.8}\right]_5^9 = [2200t]_5^9 + \frac{25}{2}\left[e^{0.8t}\right]_5^9$$
$$= 2200(9 - 5) + 12.5\left(e^{7.2} - e^4\right) \approx 24{,}860$$

13. $F = \dfrac{\pi P R^4}{8\eta l} = \dfrac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

14. If the flux remains constant, then $\dfrac{\pi P_0 R_0^4}{8\eta l} = \dfrac{\pi P R^4}{8\eta l} \Rightarrow P_0 R_0^4 = P R^4 \Rightarrow \dfrac{P}{P_0} = \left(\dfrac{R_0}{R}\right)^4$.

$R = \frac{3}{4}R_0 \Rightarrow \dfrac{P}{P_0} = \left(\dfrac{R_0}{\frac{3}{4}R_0}\right)^4 \Rightarrow P = P_0\left(\frac{4}{3}\right)^4 \approx 3.1605 P_0 > 3P_0$; that is, the blood pressure is more than tripled.

15. $\int_0^{12} c(t)\,dt = \int_0^{12} \frac{1}{4}t(12 - t)\,dt = \int_0^{12}\left(3t - \frac{1}{4}t^2\right) dt = \left[\frac{3}{2}t^2 - \frac{1}{12}t^3\right]_0^{12} = (216 - 144) = 72\ \text{mg}\cdot\text{s/L}$.

Thus, the cardiac output is $F = \dfrac{A}{\int_0^{12} c(t)\,dt} = \dfrac{8\ \text{mg}}{72\ \text{mg}\cdot\text{s/L}} = \dfrac{1}{9}\ \text{L/s} = \dfrac{60}{9}\ \text{L/min}$.

16. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = 2$.

$$\int_0^{20} c(t)\,dt \approx \tfrac{2}{3}[1(0) + 4(2.4) + 2(5.1) + 4(7.8) + 2(7.6)$$
$$+ 4(5.4) + 2(3.9) + 4(2.3) + 2(1.6) + 4(0.7) + 1(0)]$$
$$= \tfrac{2}{3}(110.8) \approx 73.87\ \text{mg}\cdot\text{s/L}$$

Therefore, $F \approx \dfrac{A}{73.87} = \dfrac{8}{73.87} \approx 0.1083$ L/s or 6.498 L/min.

8.5 Probability

1. (a) $\int_{30{,}000}^{40{,}000} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) $\int_{25{,}000}^{\infty} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)\,dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)\,dt$.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64}x\sqrt{16 - x^2} \geq 0$, so $f(x) \geq 0$ for all x.

Also, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^4 \frac{3}{64}x\sqrt{16 - x^2}\,dx = -\frac{3}{128}\int_0^4 (16 - x^2)^{1/2}(-2x)\,dx = -\frac{3}{128}\left[\frac{2}{3}(16 - x^2)^{3/2}\right]_0^4$
$$= -\tfrac{1}{64}\left[(16 - x^2)^{3/2}\right]_0^4 = -\tfrac{1}{64}(0 - 64) = 1.$$

Therefore, f is a probability density function.

(b) $P(X<2)=\int_{-\infty}^{2} f(x)\,dx=\int_0^2 \frac{3}{64}x\sqrt{16-x^2}\,dx=-\frac{3}{128}\int_0^2(16-x^2)^{1/2}(-2x)\,dx$

$=-\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^2=-\frac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2=-\frac{1}{64}(12^{3/2}-16^{3/2})$

$=\frac{1}{64}(64-12\sqrt{12})=\frac{1}{64}(64-24\sqrt{3})=1-\frac{3}{8}\sqrt{3}\approx 0.350481$

4. (a) For $0\le x\le 1$, we have $f(x)=kx^2(1-x)$, which is nonnegative if and only if $k\ge 0$. Also, $\int_{-\infty}^{\infty} f(x)\,dx=\int_0^1 kx^2(1-x)\,dx=k\int_0^1(x^2-x^3)\,dx=k\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=k/12$. Now $k/12=1 \Leftrightarrow k=12$. Therefore, f is a probability density function if and only if $k=12$.

(b) Let $k=12$.

$P(X\ge\frac{1}{2})=\int_{1/2}^{\infty} f(x)\,dx=\int_{1/2}^{1}12x^2(1-x)\,dx=\int_{1/2}^{1}(12x^2-12x^3)\,dx=\left[4x^3-3x^4\right]_{1/2}^{1}$

$=(4-3)-\left(\frac{1}{2}-\frac{3}{16}\right)=1-\frac{5}{16}=\frac{11}{16}$

(c) The mean

$\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^1 x\cdot 12x^2(1-x)\,dx=12\int_0^1(x^3-x^4)\,dx=12\left[\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^1$

$=12\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{12}{20}=\frac{3}{5}$

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x)\ge 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx=1$. Since $f(x)=0$ or $f(x)=0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{10}0.1\,dx=\left[\frac{1}{10}x\right]_0^{10}=1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x=5$.

$\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^{10}x(0.1)\,dx=\left[\frac{1}{20}x^2\right]_0^{10}=\frac{100}{20}=5$, as expected.

6. (a) As in the preceding exercise, (1) $f(x)\ge 0$ and (2) $\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{10}f(x)\,dx=\frac{1}{2}(10)(0.2)$ [area of a triangle] $=1$. So $f(x)$ is a probability density function.

(b) (i) $P(X<3)=\int_0^3 f(x)\,dx=\frac{1}{2}(3)(0.1)=\frac{3}{20}=0.15$

(ii) We first compute $P(X>8)$ and then subtract that value and our answer in (i) from 1 (the total probability).

$P(X>8)=\int_8^{10}f(x)\,dx=\frac{1}{2}(2)(0.1)=\frac{2}{20}=0.10$. So $P(3\le X\le 8)=1-0.15-0.10=0.75$.

(c) We find equations of the lines from $(0,0)$ to $(6,0.2)$ and from $(6,0.2)$ to $(10,0)$, and find that

$$f(x)=\begin{cases}\frac{1}{30}x & \text{if } 0\le x<6\\ -\frac{1}{20}x+\frac{1}{2} & \text{if } 6\le x<10\\ 0 & \text{otherwise}\end{cases}$$

$\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^6 x\left(\frac{1}{30}x\right)dx+\int_6^{10}x\left(-\frac{1}{20}x+\frac{1}{2}\right)dx=\left[\frac{1}{90}x^3\right]_0^6+\left[-\frac{1}{60}x^3+\frac{1}{4}x^2\right]_6^{10}$

$=\frac{216}{90}+\left(-\frac{1000}{60}+\frac{100}{4}\right)-\left(-\frac{216}{60}+\frac{36}{4}\right)=\frac{16}{3}=5.\overline{3}$

7. We need to find m so that $\int_m^{\infty} f(t)\,dt=\frac{1}{2} \Rightarrow \lim\limits_{x\to\infty}\int_m^x \frac{1}{5}e^{-t/5}\,dt=\frac{1}{2} \Rightarrow \lim\limits_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x=\frac{1}{2} \Rightarrow$ $(-1)(0-e^{-m/5})=\frac{1}{2} \Rightarrow e^{-m/5}=\frac{1}{2} \Rightarrow -m/5=\ln\frac{1}{2} \Rightarrow m=-5\ln\frac{1}{2}=5\ln 2\approx 3.47$ min.

8. (a) $\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$

(i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t/1000}\,dt = \lim\limits_{x\to\infty} \left[-e^{-t/1000}\right]_{800}^{x} = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty} \int_m^x \frac{1}{1000}e^{-t/1000}\,dt = \frac{1}{2} \Rightarrow$
$\lim\limits_{x\to\infty} \left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln\frac{1}{2} \Rightarrow$
$m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

9. We use an exponential density function with $\mu = 2.5$ min.

(a) $P(X > 4) = \int_4^\infty f(t)\,dt = \lim\limits_{x\to\infty} \int_4^x \frac{1}{2.5}e^{-t/2.5}\,dt = \lim\limits_{x\to\infty} \left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$

(b) $P(0 \leq X \leq 2) = \int_0^2 f(t)\,dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$

(c) We need to find a value a so that $P(X \geq a) = 0.02$, or, equivalently, $P(0 \leq X \leq a) = 0.98 \Leftrightarrow$
$\int_0^a f(t)\,dt = 0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a = 0.98 \Leftrightarrow -e^{-a/2.5} + 1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow$
$-a/2.5 = \ln 0.02 \Leftrightarrow a = -2.5\ln\frac{1}{50} = 2.5\ln 50 \approx 9.78 \text{ min} \approx 10$ min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

10. (a) With $\mu = 69$ and $\sigma = 2.8$, we have $P(65 \leq X \leq 73) = \displaystyle\int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x-69)^2}{2\cdot 2.8^2}\right) dx \approx 0.847$ (using a calculator or computer to estimate the integral).

(b) $P(X > 6 \text{ feet}) = P(X > 72 \text{ inches}) = 1 - P(0 \leq X \leq 72) \approx 1 - 0.858 = 0.142$, so 14.2% of the adult male population is more than 6 feet tall.

11. $P(X \geq 10) = \displaystyle\int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right) dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X \geq 10) \approx \displaystyle\int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right) dx \approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.
Note: We can't evaluate $1 - P(0 \leq X \leq 10)$ for this problem since a significant amount of area lies to the left of $X = 0$.

12. (a) $P(0 \leq X \leq 480) = \displaystyle\int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2\cdot 12^2}\right) dx \approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0 \leq X < 500) = 0.05$. Using our calculator or computer to find $P(0 \leq X \leq 500)$ for various values of μ, we find that if $\mu = 519.73$, $P = 0.05007$; and if $\mu = 519.74$, $P = 0.04998$. So a good target weight is at least 519.74 g.

13. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \displaystyle\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$. Substituting $t = \dfrac{x-\mu}{\sigma}$ and $dt = \dfrac{1}{\sigma}\,dx$ gives us

$$\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\,dt) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-t^2/2}\,dt \approx 0.9545$$

14. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$ where $c = 1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}\,dx = \left(e^{bx}/b^2\right)(bx-1)$

(2): $\int x^2e^{bx}\,dx = \left(e^{bx}/b^3\right)\left(b^2x^2 - 2bx + 2\right)$

Now

$$\begin{aligned}\sigma^2 &= \textstyle\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0}(x-\mu)^2 f(x)\,dx + \int_0^{\infty}(x-\mu)^2 f(x)\,dx \\ &= 0 + \lim_{t\to\infty} c\textstyle\int_0^t (x-\mu)^2 e^{-cx}\,dx = c \cdot \lim_{t\to\infty}\int_0^t \left(x^2e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}\right)dx\end{aligned}$$

Next we use (2) and (1) with $b = -c$ to get

$$\sigma^2 = c\lim_{t\to\infty}\left[-\frac{e^{-cx}}{c^3}\left(c^2x^2 + 2cx + 2\right) - 2\mu\frac{e^{-cx}}{c^2}(-cx-1) + \mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu = 1/c$, we get

$$\sigma^2 = c\left[0 - \left(-\frac{2}{c^3} + \frac{2}{c}\cdot\frac{1}{c^2} + \frac{1}{c^2}\cdot\frac{1}{-c}\right)\right] = c\left(\frac{1}{c^3}\right) = \frac{1}{c^2} \quad\Rightarrow\quad \sigma = \frac{1}{c} = \mu$$

15. (a) First $p(r) = \dfrac{4}{a_0^3}r^2e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty}\frac{4}{a_0^3}r^2e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^2e^{-2r/a_0}\,dr$$

By using parts, tables, or a CAS [or as in Exercise 14] , we find that $\int x^2e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$. (*)

Next, we use (*) (with $b = -2/a_0$) and l'Hospital's Rule to get $\dfrac{4}{a_0^3}\left[\dfrac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\dfrac{4}{a_0^3}\lim\limits_{r\to\infty}\dfrac{r^2}{e^{2r/a_0}} = \dfrac{4}{a_0^3}\lim\limits_{r\to\infty}\dfrac{2r}{(2/a_0)e^{2r/a_0}} = \dfrac{2}{a_0^2}\lim\limits_{r\to\infty}\dfrac{2}{(2/a_0)e^{2r/a_0}} = 0.$

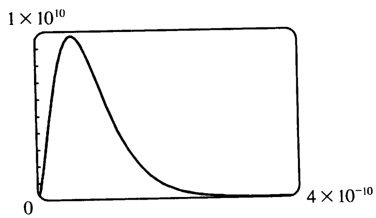
To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3}\left[r^2e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3}e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

$p'(r) = 0 \;\Leftrightarrow\; r = 0$ or $1 = \dfrac{r}{a_0} \;\Leftrightarrow\; r = a_0$ $[a_0 \approx 5.59 \times 10^{-11}$ m]. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3}a_0^2e^{-2a_0/a_0} = \frac{4}{a_0}e^{-2} \approx 9{,}684{,}098{,}979$$



With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.

(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds \;\Rightarrow\; P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$. Using (*) from part (a) (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64+16+2) - 1(2)\right]$$
$$= -\tfrac{1}{2}\left(82e^{-8} - 2\right) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} rp(r)\,dr = \frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^3 e^{-2r/a_0}\,dr$. Integrating by parts three times or using a CAS, we find that

$\int x^3 e^{bx}\,dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get

$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0$.

8 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 8.1.

   (b) See Equation 8.1.2.

   (c) See Equation 8.1.4.

2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$

   (b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y\,\sqrt{1 + [g'(y)]^2}\,dy$.

   (c) $S = \int_a^b 2\pi x\,\sqrt{1 + [f'(x)]^2}\,dx$ or $S = \int_c^d 2\pi g(y)\,\sqrt{1 + [g'(y)]^2}\,dy$

3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\,dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.

4. (a) The center of mass is the point at which the plate balances horizontally.

   (b) See Equations 8.3.8.

5. If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. See Figure 3 in Section 8.4, and the discussion which precedes it.

7. (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 8.4.

   (b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 8.4.

8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\,dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\,dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{100} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 100 pounds.

(b) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} xf(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

10. See the discussion near Equation 3 in Section 8.5.

EXERCISES

1. $y = \frac{1}{6}(x^2+4)^{3/2} \quad\Rightarrow\quad dy/dx = \frac{1}{4}(x^2+4)^{1/2}(2x) \quad\Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left[\tfrac{1}{2}x(x^2+4)^{1/2}\right]^2 = 1 + \tfrac{1}{4}x^2(x^2+4) = \tfrac{1}{4}x^4 + x^2 + 1 = \left(\tfrac{1}{2}x^2+1\right)^2.$$

Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2+1\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}x^2+1\right)dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}$.

2. $y = 2\ln\left(\sin\frac{1}{2}x\right) \quad\Rightarrow\quad \dfrac{dy}{dx} = 2\cdot\dfrac{1}{\sin\left(\frac{1}{2}x\right)}\cdot\cos\left(\frac{1}{2}x\right)\cdot\frac{1}{2} = \cot\left(\frac{1}{2}x\right) \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \cot^2\left(\frac{1}{2}x\right) = \csc^2\left(\frac{1}{2}x\right)$. Thus,

$$L = \int_{\pi/3}^{\pi} \sqrt{\csc^2\left(\tfrac{1}{2}x\right)}\,dx = \int_{\pi/3}^{\pi} \left|\csc\left(\tfrac{1}{2}x\right)\right|dx = \int_{\pi/3}^{\pi} \csc\left(\tfrac{1}{2}x\right)dx = \int_{\pi/6}^{\pi/2} \csc u\,(2\,du) \quad \begin{bmatrix} u = \frac{1}{2}x, \\ du = \frac{1}{2}\,dx \end{bmatrix}$$

$$= 2\Big[\ln|\csc u - \cot u|\Big]_{\pi/6}^{\pi/2} = 2\left[\ln\left|\csc\tfrac{\pi}{2} - \cot\tfrac{\pi}{2}\right| - \ln\left|\csc\tfrac{\pi}{6} - \cot\tfrac{\pi}{6}\right|\right]$$

$$= 2\left[\ln|1-0| - \ln\left|2-\sqrt{3}\right|\right] = -2\ln\left(2-\sqrt{3}\right) \approx 2.63$$

3. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2$.

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}$.

(b) $S = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right)dx = 2\pi\int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right)dx = 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$

$= 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

4. (a) $y = x^2 \quad\Rightarrow\quad 1 + (y')^2 = 1 + 4x^2 \quad\Rightarrow$

$$S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx = \int_1^5 \tfrac{\pi}{4}\sqrt{u}\,du \;\; [u = 1+4x^2] \;= \tfrac{\pi}{6}\left[u^{3/2}\right]_1^5 = \tfrac{\pi}{6}\left(5^{3/2} - 1\right)$$

(b) $y = x^2 \quad\Rightarrow\quad 1 + (y')^2 = 1 + 4x^2$. So

$$S = 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx = 2\pi\int_0^2 \tfrac{1}{4}u^2\sqrt{1+u^2}\,\tfrac{1}{2}\,du \quad [u = 2x] \quad = \tfrac{\pi}{4}\int_0^2 u^2\sqrt{1+u^2}\,du$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{8}u(1+2u^2)\sqrt{1+u^2} - \tfrac{1}{8}\ln\left|u + \sqrt{1+u^2}\right|\right]_0^2 \qquad [u = \tan\theta \text{ or use Formula 22}]$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{4}(9)\sqrt{5} - \tfrac{1}{8}\ln\left(2+\sqrt{5}\right) - 0\right] = \tfrac{\pi}{32}\left[18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right]$$

5. $y = e^{-x^2} \quad \Rightarrow \quad dy/dx = -2xe^{-x^2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}$.

Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)]$$
$$\approx 3.292287$$

6. $S = \int_0^3 2\pi y\,ds = \int_0^3 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}\,dx$. Let $g(x) = 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$S = \int_0^3 g(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}[g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)]$$
$$\approx 6.648327$$

7. $y = \displaystyle\int_1^x \sqrt{\sqrt{t} - 1}\,dt \quad \Rightarrow \quad dy/dx = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}$.

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}$.

8. $S = \displaystyle\int_1^{16} 2\pi x\,ds = 2\pi \int_1^{16} x \cdot x^{1/4}\,dx = 2\pi \int_1^{16} x^{5/4}\,dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512 - 1) = \frac{4088\pi}{9}$

9. As in Example 1 of Section 8.3, $\dfrac{a}{2 - x} = \dfrac{1}{2} \quad \Rightarrow \quad 2a = 2 - x$ and

$w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x$. Thus,

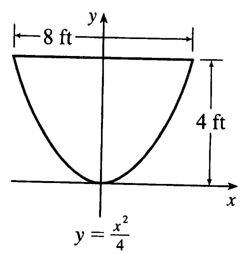
$F = \int_0^2 \rho g x(5 - x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \;\; [\rho g = \delta] \approx \frac{22}{3} \cdot 62.5 \approx 458$ lb.

10. $F = \int_0^4 \delta(4 - y)2\left(2\sqrt{y}\right)dy = 4\delta\int_0^4\left(4y^{1/2} - y^{3/2}\right)dy$

$= 4\delta\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = 4\delta\left(\frac{64}{3} - \frac{64}{5}\right)$

$= 256\delta\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{512}{15}\delta$

≈ 2133.3 lb $\quad [\delta \approx 62.5 \text{ lb/ft}^3]$



11. $A = \int_{-2}^1 \left[(4 - x^2) - (x + 2)\right]dx = \int_{-2}^1 (2 - x - x^2)\,dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1$
$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2} \quad \Rightarrow$

$$\overline{x} = A^{-1}\int_{-2}^1 x(2 - x - x^2)dx = \frac{2}{9}\int_{-2}^1 (2x - x^2 - x^3)\,dx = \frac{2}{9}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-2}^1$$
$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - 4\right)\right] = -\frac{1}{2}$$

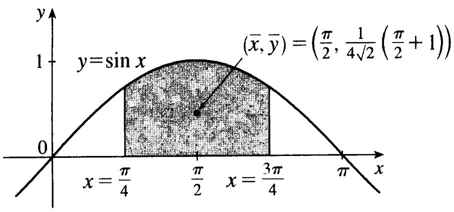
and

$$\overline{y} = A^{-1}\int_{-2}^1 \frac{1}{2}\left[\left(4 - x^2\right)^2 - (x + 2)^2\right]dx = \frac{1}{9}\int_{-2}^1 \left(x^4 - 9x^2 - 4x + 12\right)dx$$
$$= \frac{1}{9}\left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x\right]_{-2}^1 = \frac{1}{9}\left[\left(\frac{1}{5} - 3 - 2 + 12\right) - \left(-\frac{32}{5} + 24 - 8 - 24\right)\right] = \frac{12}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.

12. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $A = \int_{\pi/4}^{3\pi/4} \sin x\,dx = [-\cos x]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$$\begin{aligned}\overline{y} &= \frac{1}{A}\int_{\pi/4}^{3\pi/4} \tfrac{1}{2}\sin^2 x\,dx = \frac{1}{A}\int_{\pi/4}^{3\pi/4} \tfrac{1}{4}(1-\cos 2x)\,dx \\ &= \frac{1}{4\sqrt{2}}\left[x - \tfrac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4} \\ &= \frac{1}{4\sqrt{2}}\left[\tfrac{3\pi}{4} - \tfrac{1}{2}(-1) - \tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right] \\ &= \frac{1}{4\sqrt{2}}\left(\tfrac{\pi}{2}+1\right)\end{aligned}$$



Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2}+1\right)\right) \approx (1.57, 0.45)$.

13. An equation of the line passing through $(0, 0)$ and $(3, 2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2}\cdot 3\cdot 2 = 3$. Therefore, using Equations 8.3.8, $\overline{x} = \frac{1}{3}\int_0^3 x\left(\frac{2}{3}x\right)dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$, and $\overline{y} = \frac{1}{3}\int_0^3 \frac{1}{2}\left(\frac{2}{3}x\right)^2 dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(2, \frac{2}{3}\right)$.

14. Suppose first that the large rectangle were complete, so that its mass would be $6\cdot 3 = 18$. Its centroid would be $\left(1, \frac{3}{2}\right)$. The mass removed from this object to create the one being studied is 3. The centroid of the cut-out piece is $\left(\frac{3}{2}, \frac{3}{2}\right)$. Therefore, for the actual lamina, whose mass is 15, $\overline{x} = \frac{18}{15}(1) - \frac{3}{15}\left(\frac{3}{2}\right) = \frac{9}{10}$, and $\overline{y} = \frac{3}{2}$, since the lamina is symmetric about the line $y = \frac{3}{2}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{10}, \frac{3}{2}\right)$.

15. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

16. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x-axis. $\overline{x} = 0$ because of symmetry about the line $x = 0$. And by the Theorem of Pappus, $V = A(2\pi\overline{y}) \Rightarrow \frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2(2\pi\overline{y}) \Rightarrow \overline{y} = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{4}{3\pi}r\right)$.

17. $x = 100 \Rightarrow P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\begin{aligned}\text{Consumer surplus} &= \int_0^{100}[p(x) - P]\,dx = \int_0^{100}\left(2000 - 0.1x - 0.01x^2 - 1890\right)dx \\ &= \left[110x - 0.05x^2 - \tfrac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \tfrac{10{,}000}{3} \approx \$7166.67\end{aligned}$$

18. $$\begin{aligned}\int_0^{24} c(t)\,dt \approx S_{12} &= \tfrac{24-0}{12\cdot 3}[1(0) + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8) \\ &\qquad + 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 1(0)] \\ &= \tfrac{2}{3}(127.8) = 85.2\text{ mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx A/85.2 = 6/85.2 \approx 0.0704$ L/s or 4.225 L/min.

19. $f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f(x) \ge 0$ for all real numbers x and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \tfrac{\pi}{20}\sin\left(\tfrac{\pi}{10}x\right)dx = \tfrac{\pi}{20}\cdot\tfrac{10}{\pi}\left[-\cos\left(\tfrac{\pi}{10}x\right)\right]_0^{10}$$
$$= \tfrac{1}{2}(-\cos\pi + \cos 0) = \tfrac{1}{2}(1+1) = 1$$

Therefore, f is a probability density function.

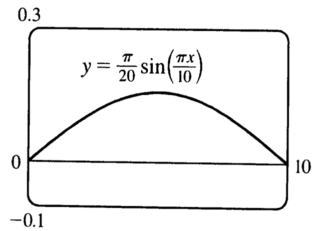
(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right)dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos\frac{2\pi}{5} + \cos 0\right)$

$\approx \frac{1}{2}(-0.309017 + 1) \approx 0.3455$

(c) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{10} \frac{\pi}{20}x\sin\left(\frac{\pi}{10}x\right)dx$

$= \int_0^{\pi} \frac{\pi}{20}\cdot\frac{10}{\pi}u(\sin u)\left(\frac{10}{\pi}\right)du \quad \left[u = \frac{\pi}{10}x,\ du = \frac{\pi}{10}\,dx\right]$

$= \frac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \frac{5}{\pi}\left[\sin u - u\cos u\right]_0^{\pi} = \frac{5}{\pi}[0 - \pi(-1)] = 5$



This answer is expected because the graph of f is symmetric about the line $x = 5$.

20. $P(250 \le X \le 280) = \displaystyle\int_{250}^{280} \frac{1}{15\sqrt{2\pi}}\exp\left(\frac{-(x-268)^2}{2\cdot 15^2}\right)dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$P(0 \le X \le 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

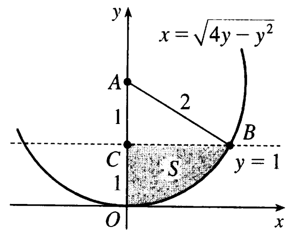
(c) We need to find m such that $P(X \ge m) = \frac{1}{2} \Rightarrow \int_m^{\infty}\frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow$

$\lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln\frac{1}{2} \Rightarrow$

$m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

PROBLEMS PLUS

1. $x^2 + y^2 \le 4y \quad \Leftrightarrow \quad x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\begin{aligned}\int_0^1 \sqrt{4y - y^2}\, dy &\overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y-y^2} + 2\cos^{-1}\left(\tfrac{2-y}{2}\right)\right]_0^1 \qquad [a = 2] \\ &= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\left(\tfrac{1}{2}\right) - 2\cos^{-1} 1 \\ &= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\end{aligned}$$

Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}(2^2)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

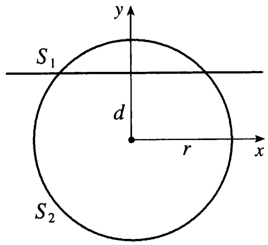
2. $y = \pm\sqrt{x^3 - x^4} \quad \Rightarrow$ The loop of the curve is symmetric about $y = 0$, and therefore $\overline{y} = 0$. At each point x where $0 \le x \le 1$, the lamina has a vertical length of $\sqrt{x^3 - x^4} - \left(-\sqrt{x^3 - x^4}\right) = 2\sqrt{x^3 - x^4}$. Therefore, $\overline{x} = \dfrac{\int_0^1 x \cdot 2\sqrt{x^3 - x^4}\, dx}{\int_0^1 2\sqrt{x^3 - x^4}\, dx} = \dfrac{\int_0^1 x\sqrt{x^3 - x^4}\, dx}{\int_0^1 \sqrt{x^3 - x^4}\, dx}$. We evaluate the integrals separately:

$$\begin{aligned}\int_0^1 x\sqrt{x^3 - x^4}\, dx &= \int_0^1 x^{5/2}\sqrt{1-x}\, dx \\ &= \int_0^{\pi/2} 2\sin^6\theta\cos\theta\sqrt{1-\sin^2\theta}\, d\theta \qquad \left[\begin{array}{c}\sin\theta = \sqrt{x},\ \cos\theta\, d\theta = dx/(2\sqrt{x}), \\ 2\sin\theta\cos\theta\, d\theta = dx\end{array}\right] \\ &= \int_0^{\pi/2} 2\sin^6\theta\cos^2\theta\, d\theta = \int_0^{\pi/2} 2\left[\tfrac{1}{2}(1-\cos 2\theta)\right]^3 \tfrac{1}{2}(1+\cos 2\theta) d\theta \\ &= \int_0^{\pi/2} \tfrac{1}{8}(1 - 2\cos 2\theta + 2\cos^3 2\theta - \cos^4 2\theta)\, d\theta \\ &= \int_0^{\pi/2} \tfrac{1}{8}\left[1 - 2\cos 2\theta + 2\cos 2\theta(1-\sin^2 2\theta) - \tfrac{1}{4}(1+\cos 4\theta)^2\right] d\theta \\ &= \tfrac{1}{8}\left[\theta - \tfrac{1}{3}\sin^3 2\theta\right]_0^{\pi/2} - \tfrac{1}{32}\int_0^{\pi/2}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right) d\theta \\ &= \tfrac{\pi}{16} - \tfrac{1}{32}\left[\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2} - \tfrac{1}{64}\int_0^{\pi/2}(1 + \cos 8\theta)\, d\theta \\ &= \tfrac{3\pi}{64} - \tfrac{1}{64}\left[\theta + \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/2} = \tfrac{5\pi}{128}\end{aligned}$$

$$\begin{aligned}\int_0^1 \sqrt{x^3 - x^4}\, dx &= \int_0^1 x^{3/2}\sqrt{1-x}\, dx = \int_0^{\pi/2} 2\sin^4\theta\cos\theta\sqrt{1-\sin^2\theta}\, d\theta \qquad [\sin\theta = \sqrt{x}\,] \\ &= \int_0^{\pi/2} 2\sin^4\theta\cos^2\theta\, d\theta = \int_0^{\pi/2} 2 \cdot \tfrac{1}{4}(1-\cos 2\theta)^2 \cdot \tfrac{1}{2}(1+\cos 2\theta)\, d\theta \\ &= \int_0^{\pi/2} \tfrac{1}{4}(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)\, d\theta \\ &= \int_0^{\pi/2} \tfrac{1}{4}\left[1 - \cos 2\theta - \tfrac{1}{2}(1+\cos 4\theta) + \cos 2\theta(1-\sin^2 2\theta)\right] d\theta \\ &= \tfrac{1}{4}\left[\tfrac{\theta}{2} - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/2} = \tfrac{\pi}{16}\end{aligned}$$

Therefore, $\overline{x} = \dfrac{5\pi/128}{\pi/16} = \dfrac{5}{8}$, and $(\overline{x}, \overline{y}) = \left(\frac{5}{8}, 0\right)$.

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy = \sqrt{\frac{r^2}{r^2 - y^2}}\,dy = \frac{r\,dy}{\sqrt{r^2 - y^2}}$$

From Formula 8.2.8 we have

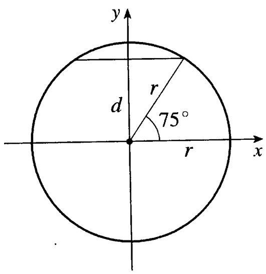
$$S_1 = \int_d^r 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_d^r 2\pi\sqrt{r^2 - y^2}\frac{r\,dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\,dy = 2\pi r(r - d)$$

Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x\sqrt{1 + (dx/dy)^2}\,dy = \int_{-r}^{d} 2\pi r\,dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 3960$ mi and $d = r\,(\sin 75^\circ) \approx 3825$ mi,

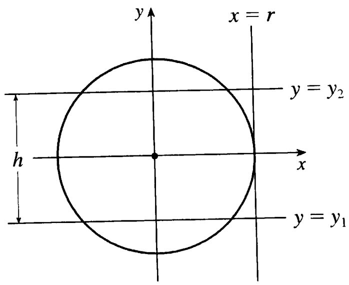
so the surface area of the Arctic Ocean is about

$2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6\ \text{mi}^2$.



(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \displaystyle\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{y_1}^{y_2} 2\pi r\,dy = 2\pi r(y_2 - y_1) = 2\pi rh.$

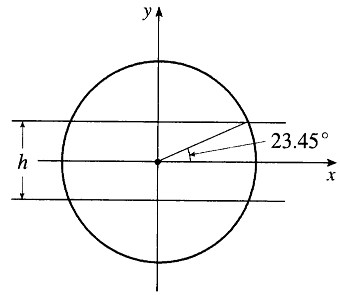
This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is



$$A = \int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{y_1}^{y_2} 2\pi r\sqrt{1 + 0^2}\,dy$$

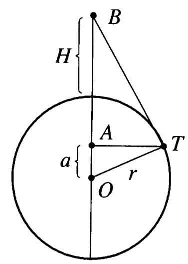
$$= 2\pi ry\big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi rh$$

(d) $h = 2r \sin 23.45^\circ \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7$ mi^2.



4. (a) Since the right triangles OAT and OTB are similar, we have

$\dfrac{r+H}{r} = \dfrac{r}{a} \quad\Rightarrow\quad a = \dfrac{r^2}{r+H}$. The surface area visible from B is



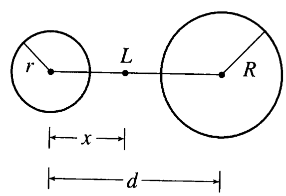
$S = \int_a^r 2\pi x \sqrt{1 + (dx/dy)^2}\, dy$. From $x^2 + y^2 = r^2$, we get

$\dfrac{d}{dy}(x^2+y^2) = \dfrac{d}{dy}(r^2) \quad\Rightarrow\quad 2x\dfrac{dx}{dy} + 2y = 0 \quad\Rightarrow$

$\dfrac{dx}{dy} = -\dfrac{y}{x}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{x^2+y^2}{x^2} = \dfrac{r^2}{x^2}$. Thus,

$$S = \int_a^r 2\pi x \cdot \frac{r}{x}\, dy = 2\pi r(r-a) = 2\pi r\left(r - \frac{r^2}{r+H}\right) = 2\pi r^2\left(1 - \frac{r}{r+H}\right) = 2\pi r^2 \cdot \frac{H}{r+H} = \frac{2\pi r^2 H}{r+H}.$$

(b) Assume $R \geq r$. If a light is placed at point L, at a distance x from the center of the sphere of radius r, then from part (a) we find that the total illuminated area A on the two spheres is [with $r + H = x$ and $r + H = d - x$].



$A(x) = \dfrac{2\pi r^2(x-r)}{x} + \dfrac{2\pi R^2(d-x-R)}{d-x}$ $[r \leq x \leq d-R]$. $\dfrac{A(x)}{2\pi} = r^2\left(1 - \dfrac{r}{x}\right) + R^2\left(1 - \dfrac{R}{d-x}\right)$, so

$A'(x) = 0 \quad\Leftrightarrow\quad 0 = r^2 \cdot \dfrac{r}{x^2} + R^2 \cdot \dfrac{-R}{(d-x)^2} \quad\Leftrightarrow\quad \dfrac{r^3}{x^2} = \dfrac{R^3}{(d-x)^2} \quad\Leftrightarrow\quad \dfrac{(d-x)^2}{x^2} = \dfrac{R^3}{r^3} \quad\Leftrightarrow$

$\left(\dfrac{d}{x} - 1\right)^2 = \left(\dfrac{R}{r}\right)^3 \quad\Rightarrow\quad \dfrac{d}{x} - 1 = \left(\dfrac{R}{r}\right)^{3/2} \quad\Leftrightarrow\quad \dfrac{d}{x} = 1 + \left(\dfrac{R}{r}\right)^{3/2} \quad\Leftrightarrow\quad x = x^* = \dfrac{d}{1 + (R/r)^{3/2}}.$

Now $A'(x) = 2\pi\left(\dfrac{r^3}{x^2} - \dfrac{R^3}{(d-x)^2}\right) \quad\Rightarrow\quad A''(x) = 2\pi\left(-\dfrac{2r^3}{x^3} - \dfrac{2R^3}{(d-x)^3}\right)$ and $A''(x^*) < 0$, so we have a local maximum at $x = x^*$.

However, x^* may not be an allowable value of x—we must show that x^* is between r and $d - R$.

(1) $x^* \geq r \quad\Leftrightarrow\quad \dfrac{d}{1 + (R/r)^{3/2}} \geq r \quad\Leftrightarrow\quad d \geq r + R\sqrt{R/r}$

(2) $x^* \le d - R \;\Leftrightarrow\; \dfrac{d}{1+(R/r)^{3/2}} \le d - R \;\Leftrightarrow\; d \le d - R + d\left(\dfrac{R}{r}\right)^{3/2} - R\left(\dfrac{R}{r}\right)^{3/2} \;\Leftrightarrow$

$R + R\left(\dfrac{R}{r}\right)^{3/2} \le d\left(\dfrac{R}{r}\right)^{3/2} \;\Leftrightarrow\; d \ge \dfrac{R}{(R/r)^{3/2}} + R = R + r\sqrt{r/R}$, but

$R + r\sqrt{r/R} \le R + r$, and since $d > r + R$ [given], we conclude that $x^* \le d - R$.

Thus, from (1) and (2), x^* is not an allowable value of x if $d < r + R\sqrt{R/r}$.

So A may have a maximum at $x = r$, x^*, or $d - R$.

$$A(r) = \frac{2\pi R^2(d-r-R)}{d-r} \quad \text{and} \quad A(d-R) = \frac{2\pi r^2(d-r-R)}{d-R}$$

$A(r) > A(d-R) \;\Leftrightarrow\; \dfrac{R^2}{d-r} > \dfrac{r^2}{d-R} \;\Leftrightarrow\; R^2(d-R) > r^2(d-r) \;\Leftrightarrow\; R^2 d - R^3 > r^2 d - r^3 \;\Leftrightarrow$

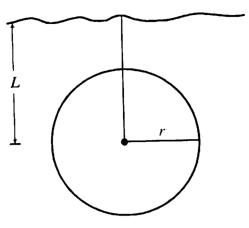
$R^2 d - r^2 d > R^3 - r^3 \;\Leftrightarrow\; d(R-r)(R+r) > (R-r)(R^2 + Rr + r^2) \;\Leftrightarrow$

$d > (R^2 + Rr + r^2)/(R+r) \;\Leftrightarrow\; d > [(R+r)^2 - Rr]/(R+r) \;\Leftrightarrow\; d > R + r - Rr/(R+r)$. Now $R + r - Rr/(R+r) < R + r$, and we know that $d > R + r$, so we conclude that $A(r) > A(d-R)$.

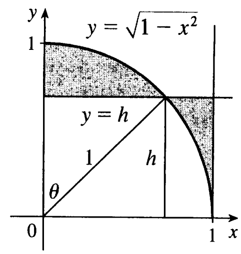
In conclusion, A has an absolute maximum at $x = x^*$ provided $d \ge r + R\sqrt{R/r}$; otherwise, A has its maximum at $x = r$.

5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

(b)



$$\begin{aligned}
F &= \int_{-r}^{r} P(L+x)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= \int_{-r}^{r}\left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\,dz\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= P_0\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 gH\int_{-r}^{r}\left(e^{(L+x)/H} - 1\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0 - \rho_0 gH)\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 gH\int_{-r}^{r} e^{(L+x)/H}\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2-x^2}\,dx
\end{aligned}$$



6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. The equation of the circle in the first quadrant is $y = \sqrt{1-x^2}$, so if the equation of the line is $y = h$, then the circle and the line intersect where $h = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-h^2}$. So the shaded area is

$$A = \int_0^{\sqrt{1-h^2}} \left(\sqrt{1-x^2} - h\right) dx + \int_{\sqrt{1-h^2}}^1 \left(h - \sqrt{1-x^2}\right) dx$$

$$\overset{\star}{=} \left[-hx\right]_0^{\sqrt{1-h^2}} + \left[hx\right]_{\sqrt{1-h^2}}^1 + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx$$

$$= -h\sqrt{1-h^2} + h - h\sqrt{1-h^2} + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx$$

$$= h\left(1 - 2\sqrt{1-h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx$$

Note that at ($\star$), we reversed the limits of integration and changed the sign in the last integral. We are interested in the minimum of

$A(h) = h\left(1 - 2\sqrt{1-h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx$, so we find dA/dh using FTC1 and the Chain Rule:

$$\frac{dA}{dh} = h\left(-2\frac{-h}{\sqrt{1-h^2}}\right) + \left(1 - 2\sqrt{1-h^2}\right) + 2\left[\sqrt{1 - \left(\sqrt{1-h^2}\right)^2}\right] \frac{d}{dh}\left(\sqrt{1-h^2}\right)$$

$$= \frac{1}{\sqrt{1-h^2}}\left[2h^2 + \sqrt{1-h^2} - 2\left(1-h^2\right)\right] + 2h\frac{-h}{\sqrt{1-h^2}}$$

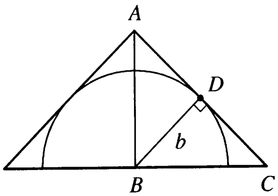
$$= \frac{1}{\sqrt{1-h^2}}\left[\sqrt{1-h^2} - 2\left(1-h^2\right)\right]$$

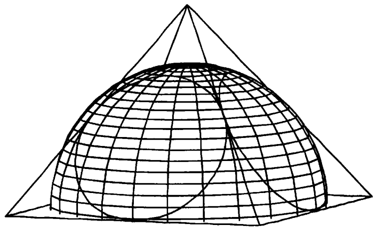
This is 0 when $\sqrt{1-h^2} - 2\left(1-h^2\right) = 0 \Leftrightarrow u - 2u^2 = 0$ (where $u = \sqrt{1-h^2}$) $\Leftrightarrow u = 0$ or $\frac{1}{2} \Leftrightarrow h = 1$ or $\frac{\sqrt{3}}{2}$. By the First Derivative Test, $h = \frac{\sqrt{3}}{2}$ represents a minimum for $A(h)$, since $A'(h) = 1 - \dfrac{2}{\sqrt{1-h^2}}$ goes from negative to positive at $h = \frac{\sqrt{3}}{2}$.

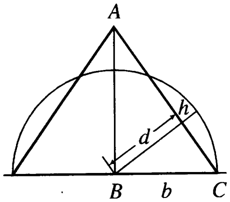
Another method: Use FTC2 to evaluate all of the integrals before differentiating.

Note: Another strategy is to use the angle θ as the variable (see diagram above) and show that $A = \theta + \cos\theta - \frac{\pi}{4} - \frac{1}{2}\sin 2\theta$, which is minimized when $\theta = \frac{\pi}{6}$.

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}b$.







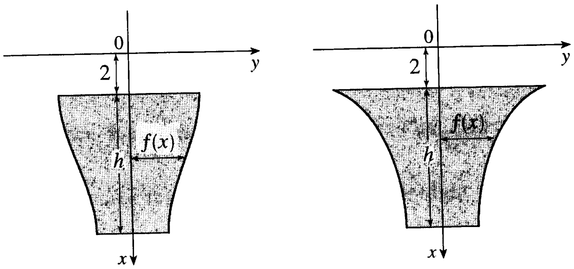
We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.49 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\,b}{\sqrt{b^2 + \left(\sqrt{2}\,b\right)^2}} \quad \Rightarrow \quad d = \frac{\sqrt{2}\,b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\,b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3 - \sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 6.2.49 with $r = b$, we find that the volume of each of the caps is

$\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3 \cdot 3}b\right) = \frac{15 - 6\sqrt{6}}{9} \cdot \frac{6 + \sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

8. Orient the positive x-axis as in the figure.



Suppose that the plate has height h and is symmetric about the x-axis. At depth x below the water ($2 \le x \le 2+h$), let the width of the plate be $2f(x)$. Now each of the n horizontal strips has height h/n and the ith strip ($1 \le i \le n$) goes from $x = 2 + \left(\frac{i-1}{n}\right)h$ to $x = 2 + \left(\frac{i}{n}\right)h$. The hydrostatic force on the ith strip is

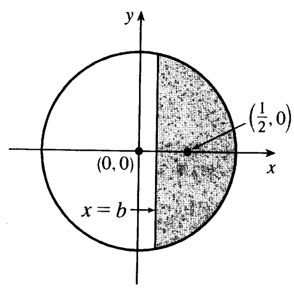
$F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5x[2f(x)]dx$. If we now let $x[2f(x)] = k$ (a constant) so that $f(x) = k/(2x)$, then

$$F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5k\,dx = 62.5k\,[x]_{2+[(i-1)/n]h}^{2+(i/n)h}$$

$$= 62.5k\left[\left(2+\tfrac{i}{n}h\right) - \left(2+\tfrac{i-1}{n}h\right)\right] = 62.5k\left(\tfrac{h}{n}\right)$$

So the hydrostatic force on the ith strip is independent of i, that is, the force on each strip is the same. So the plate can be shaped as shown in the figure. (In fact, the required condition is satisfied whenever the plate has width C/x at depth x, for some constant C. Many shapes are possible.)

9. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have



$\frac{1}{2} = \overline{x} = \dfrac{\int_b^1 x \cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the numerator gives us

$-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0 - \left(1-b^2\right)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$. Using Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_b^1 = \left(0 + \frac{\pi}{2}\right) - \left(b\sqrt{1-b^2} + \sin^{-1}b\right)$. Thus, we have

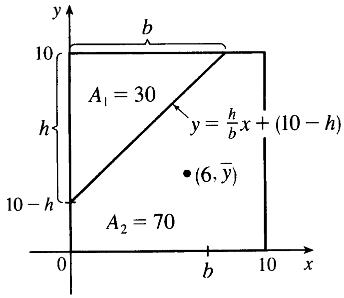
$\frac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}$, or, equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

10. $A_1 = 30 \;\Rightarrow\; \frac{1}{2}bh = 30 \;\Rightarrow\; bh = 60$.

$$\overline{x} = 6 \;\Rightarrow\; \frac{1}{A_2}\int_0^{10} xf(x)\,dx = 6 \;\Rightarrow\; \int_0^b x\left(\frac{h}{b}x + 10 - h\right)dx + \int_b^{10} x(10)\,dx = 6(70) \;\Rightarrow$$

$$\int_0^b \left(\frac{h}{b}x^2 + 10x - hx\right)dx + 10 \cdot \tfrac{1}{2}\left[x^2\right]_b^{10} = 420 \;\Rightarrow\; \left[\frac{h}{3b}x^3 + 5x^2 - \frac{h}{2}x^2\right]_0^b + 5(100 - b^2) = 420 \;\Rightarrow$$

$\frac{1}{3}hb^2 + 5b^2 - \frac{1}{2}hb^2 + 500 - 5b^2 = 420 \quad \Rightarrow \quad 80 = \frac{1}{6}hb^2 \quad \Rightarrow \quad 480 = (hb)b \quad \Rightarrow \quad 480 = 60b \quad \Rightarrow \quad b = 8.$



So $h = \frac{60}{8} = \frac{15}{2}$ and an equation of the line is $y = \dfrac{15/2}{8}x + \left(10 - \dfrac{15}{2}\right) = \dfrac{15}{16}x + \dfrac{5}{2}$. Now

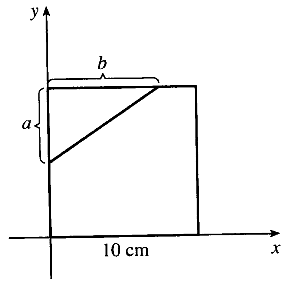
$$\overline{y} = \frac{1}{A_2}\int_0^{10} \frac{1}{2}[f(x)]^2\,dx = \frac{1}{70\cdot 2}\left[\int_0^8 \left(\frac{15}{16}x + \frac{5}{2}\right)^2 dx + \int_8^{10} (10)^2\,dx\right]$$

$$= \tfrac{1}{140}\left[\int_0^8 \left(\tfrac{225}{256}x^2 + \tfrac{75}{16}x + \tfrac{25}{4}\right)dx + 100(10-8)\right] = \tfrac{1}{140}\left(\left[\tfrac{225}{768}x^3 + \tfrac{75}{32}x^2 + \tfrac{25}{4}x\right]_0^8 + 200\right)$$

$$= \tfrac{1}{140}(150 + 150 + 50 + 200) = \tfrac{550}{140} = \tfrac{55}{14}$$

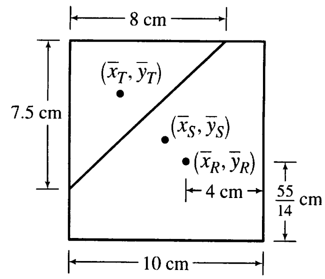
Another solution:

Assume that the right triangle cut from the square has legs a cm and b cm long as shown. The triangle has area 30 cm^2, so $\frac{1}{2}ab = 30$ and $ab = 60$. We place the square in the first quadrant of the xy-plane as shown, and we let T, R, and S denote the triangle, the remaining portion of the square, and the full square, respectively. By symmetry, the centroid of S is $(5, 5)$. By Exercise 8.3.37, the centroid of T is $\left(\dfrac{b}{3}, 10 - \dfrac{a}{3}\right)$.



We are given that the centroid of R is $(6, c)$, where c is to be determined. We take the density of the square to be 1, so that areas can be used as masses. Then T has mass $m_T = 30$, S has mass $m_S = 100$, and R has mass $m_R = m_S - m_T = 70$. As in Exercises 38 and 39 of Section 8.3, we view S as consisting of a mass m_T at the centroid $(\overline{x}_T, \overline{y}_T)$ of T and a mass R at the centroid $(\overline{x}_R, \overline{y}_R)$ of R. Then $\overline{x}_S = \dfrac{m_T\overline{x}_T + m_R\overline{x}_R}{m_T + m_R}$ and $\overline{y}_S = \dfrac{m_T\overline{y}_T + m_R\overline{y}_R}{m_T + m_R}$; that is, $5 = \dfrac{30(b/3) + 70(6)}{100}$ and $5 = \dfrac{30(10 - a/3) + 70c}{100}$.

Solving the first equation for b, we get $b = 8$ cm. Since $ab = 60\text{ cm}^2$, it follows that $a = \frac{60}{8} = 7.5$ cm. Now the second equation says that $70c = 200 + 10a$, so $7c = 20 + a = \frac{55}{2}$ and $c = \frac{55}{14} = 3.9\overline{285714}$ cm. The solution is depicted in the figure.



11. If $h = L$, then

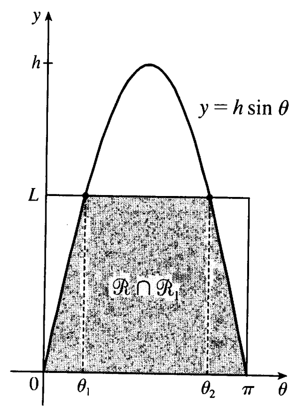
$$P = \frac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{\pi} = \frac{-(-1)+1}{\pi} = \frac{2}{\pi}$$

If $h = L/2$, then

$$P = \frac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi \frac{1}{2}L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$$

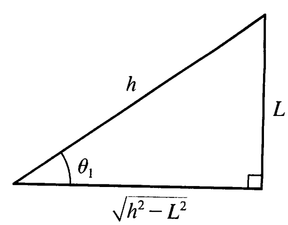
12. (a) The total set of possibilities can be identified with the rectangular region $\mathfrak{R} = \{(\theta, y) \mid 0 \le y < L, 0 \le \theta < \pi\}$. Even when $h > L$, the needle intersects at least one line if and only if $y \le h\sin\theta$. Let $\mathfrak{R}_1 = \{(\theta, y) \mid 0 \le y \le h\sin\theta, 0 \le \theta < \pi\}$. When $h \le L$, $\mathfrak{R}_1$ is contained in $\mathfrak{R}$, but that is no longer true when $h > L$. Thus, the probability that the needle intersects a line becomes

$$P = \frac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_1)}{\text{area}(\mathfrak{R})} = \frac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_1)}{\pi L}$$



When $h > L$, the curve $y = h\sin\theta$ intersects the line $y = L$ twice—at $\left(\sin^{-1}(L/h), L\right)$ and at $\left(\pi - \sin^{-1}(L/h), L\right)$. Set $\theta_1 = \sin^{-1}(L/h)$ and $\theta_2 = \pi - \theta_1$. Then

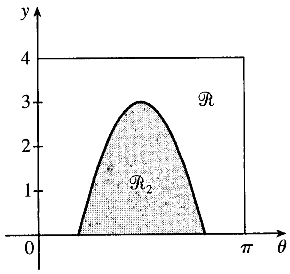
$$\begin{aligned}
\text{area}(\mathfrak{R} \cap \mathfrak{R}_1) &= \int_0^{\theta_1} h\sin\theta\,d\theta + \int_{\theta_1}^{\theta_2} L\,d\theta + \int_{\theta_2}^{\pi} h\sin\theta\,d\theta \\
&= 2\int_0^{\theta_1} h\sin\theta\,d\theta + L(\theta_2 - \theta_1) = 2h\,[-\cos\theta]_0^{\theta_1} + L(\pi - 2\theta_1) \\
&= 2h(1 - \cos\theta_1) + L(\pi - 2\theta_1) \\
&= 2h\left(1 - \frac{\sqrt{h^2 - L^2}}{h}\right) + L\left[\pi - 2\sin^{-1}\left(\frac{L}{h}\right)\right] \\
&= 2h - 2\sqrt{h^2 - L^2} + \pi L - 2L\sin^{-1}\left(\frac{L}{h}\right)
\end{aligned}$$



We are told that $L = 4$ and $h = 7$, so $\text{area}(\mathfrak{R} \cap \mathfrak{R}_1) = 14 - 2\sqrt{33} + 4\pi - 8\sin^{-1}\left(\frac{4}{7}\right) \approx 10.21128$ and

$P = \frac{1}{4\pi}\,\text{area}(\mathfrak{R} \cap \mathfrak{R}_1) \approx 0.812588$. (By comparison, $P = \frac{2}{\pi} \approx 0.636620$ when $h = L$, as shown in the solution to Problem 11.)

(b) The needle intersects at least two lines when $y + L \le h \sin\theta$; that is, when $y \le h\sin\theta - L$. Set $\mathfrak{R}_2 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - L, 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least two lines is $P_2 = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_2)}{\text{area}(\mathfrak{R})} = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_2)}{\pi L}$. When $L = 4$ and $h = 7$, $\mathfrak{R}_2$ is contained in $\mathfrak{R}$ (see the figure). Thus,



$$P_2 = \frac{1}{4\pi}\,\text{area}(\mathfrak{R}_2) = \frac{1}{4\pi}\int_{\sin^{-1}(4/7)}^{\pi - \sin^{-1}(4/7)} (7\sin\theta - 4)\,d\theta = \frac{1}{4\pi} \cdot 2\int_{\sin^{-1}(4/7)}^{\pi/2} (7\sin\theta - 4)\,d\theta$$

$$= \frac{1}{2\pi}\left[-7\cos\theta - 4\theta\right]_{\sin^{-1}(4/7)}^{\pi/2} = \frac{1}{2\pi}\left[0 - 2\pi + 7\frac{\sqrt{33}}{7} + 4\sin^{-1}(4/7)\right]$$

$$= \frac{\sqrt{33} + 4\sin^{-1}(4/7) - 2\pi}{2\pi} \approx 0.301497$$

(c) The needle intersects at least three lines when $y + 2L \le h\sin\theta$: that is, when $y \le h\sin\theta - 2L$. Set $\mathfrak{R}_3 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - 2L, 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least three lines is $P_3 = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_3)}{\text{area}(\mathfrak{R})} = \dfrac{\text{area}(\mathfrak{R} \cap \mathfrak{R}_3)}{\pi L}$. (At this point, the generalization to P_n, n any positive integer, should be clear.) Under the given assumption,

$$P_3 = \frac{1}{\pi L}\,\text{area}(\mathfrak{R}_3) = \frac{1}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi - \sin^{-1}(2L/h)} (h\sin\theta - 2L)\,d\theta$$

$$= \frac{2}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi/2} (h\sin\theta - 2L)\,d\theta$$

$$= \frac{2}{\pi L}\left[-h\cos\theta - 2L\theta\right]_{\sin^{-1}(2L/h)}^{\pi/2}$$

$$= \frac{2}{\pi L}\left[-\pi L + \sqrt{h^2 - 4L^2} + 2L\sin^{-1}(2L/h)\right]$$

Note that the probability that a needle touches exactly one line is $P_1 - P_2$, the probability that it touches exactly two lines is $P_2 - P_3$, and so on.